Comp 311
Functional Programming

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Options

- Often the result of a computation is that no satisfactory value could be found
  - Lookup in a table with a key that does not exist
  - Attempting to find a path that does not exist

Scala Options

```
abstract class Option[+A] {…}

case object None extends Option[Nothing] {…}

case class Some[+A](val contained: A) extends Option[A]
{
  …
}
```

Options Are Monads!

```
abstract class Option[+A] {
  def flatMap[B](f: (A) ⇒ Option[B]): Option[B]
  def map[B](f: (A) ⇒ B): Option[B]
  def withFilter(p: (A) ⇒ Boolean):
    FilterMonadic[A, collection.Iterable[A]]
}
```

Contract Attempt 2

```
/**
 * Create a path from start to finish in G, if
 * it exists.
 */
def findRoute(start: String, end: String,
              graph: Graph):
             Option[List[String]]
```

Reduce to Backtracking Cases

```
def findRoute(start: String, end: String,
              graph: Graph): Option[List[String]] = {
  if (start == end) Some(List(end))
  else for (route <- routeFromOrigins(graph(start), end, graph))
         yield start :: route
}
```

Recursive Sub-Problems

```
def routeFromOrigins(origins: List[String], destination: String,
                     graph: Graph): Option[List[String]] = {
  origins match {
    case Nil => None
    case origin :: origins => {
      findRoute(origin, destination, graph) match {
        case None => routeFromOrigins(origins, destination,graph)
        case Some(route) => Some(route)
      }
    }
  }
}
```

Termination

- `routeFromOrigins` is structurally recursive:
  - terminates provided that findRoute terminates
- `findRoute` terminates only if graph is acyclic

Accumulating Knowledge

Accumulating Knowledge

- Remember visited nodes to prevent infinite regress
- Pass this to recursive calls via “accumulator”

Reduce to Backtracking

```
def findRoute(start: String, end: String, graph: Graph,
              visited: List[String] = Nil):
Option[List[String]] = {
  if (start == end) Some(List(end))
  else if (visited contains start) None
  else for (route <- routeFromOrigins(graph(start), end, graph,
                                      start :: visited))
         yield start :: route
}
```

Reduce to Backtracking

```
def routeFromOrigins(origins: List[String], destination: String,
                     graph: Graph, visited: List[String] = Nil):
Option[List[String]] = {
  origins match {
    case Nil => None
    case origin :: origins => {
      findRoute(origin, destination, graph, visited) match {
        case None => routeFromOrigins(origins, destination,
                                      graph, origin :: visited)
        case Some(route) => Some(route)
      }
    }
  }
}
```

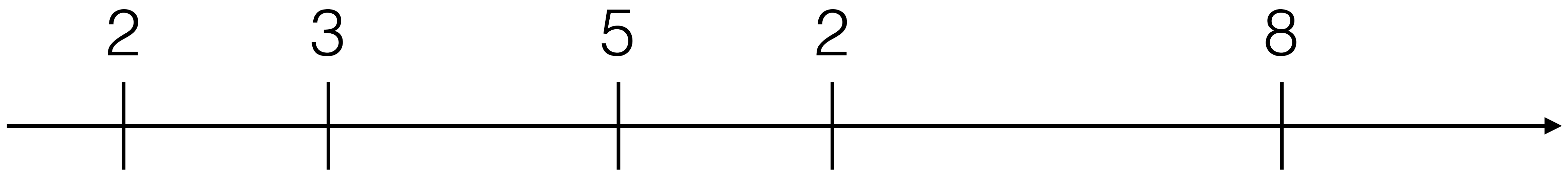
Accumulators

- accumulator parameter allows us to “remember” knowledge from one recursive call to another
  - Often essential for correctness in generative recursion
  - Also useful for saving space in structural recursion

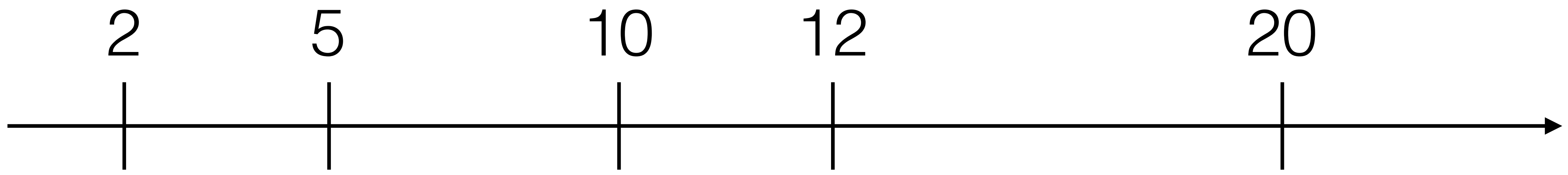
Accumulators for Structural Recursion

- Let us define a function `fromOrigin`, which:
  - Takes a list of `Int` values, with each value denoting a relative distance to the point to its left
  - Returns a list of `Int` values denoting the absolute distances to the origin

Accumulators for Structural Recursion



becomes



Defining fromOrigin

```
def fromOrigin[T](xs: List[T]) = {
  xs match {
    case Nil => Nil
    case x :: xs => x :: (fromOrigin {xs} map {_+x})
  }
}
```

Defining fromOrigin

```
def fromOrigin (xs: List[Int]): List[Int] = {
  xs match {
    case Nil => Nil
    case x :: xs =>
      x :: (for (y <- fromOrigin(xs)) yield {y+x})
  }
}
```

How many steps does it take to compute an application of fromOrigin, in comparison to the length of the list?

cost of fromOrigin

```
fromOrigin(List(2,3,5,2,8)) ↦
    List(2,3,5,2,8) match {
      case Empty => Empty
      case x :: xs => x :: (fromOrigin {xs} map {_+x})
    }  ↦
2 :: (fromOrigin(List(3,5,2,8)) map (_+2)) ↦*
2 :: (3 :: (fromOrigin(List(5,2,8) map (_+3))) map(_+2)) ↦*
2 :: (3 :: (List(5, 7, 15) map (_+3))) map(_+2)) ↦*
2 :: (3 :: (List(8, 10, 18)) map(_+2)) ↦*
2 :: (List(5, 10, 12, 20)) ↦*
List(2, 5, 10, 12, 20)
```

The cost of fromOrigin

- Each recursive call map over the argument list
  - which takes *n* steps for a list of length *n*

$$\sum_{i=1}^{n} i = \frac{(n)(1+n)}{2} = O(n^2)$$

Big O Notation

- We say:

$$f(x) = O(g(x)) \ \texttt{as} \ x \to \infty$$

- To mean that there is a constant *k* and some value x_0 such that

$$|f(x)| \leq k|g(x)| \ \texttt{for all} \ x \geq x_0$$

Big O Notation

- Typically the part:

$$\texttt{as} \;\; x \to \infty$$

- is implicit

- Effectively, we are defining equivalence classes of functions

Accumulating Distance to the Origin

- We could reduce the time taken by instead accumulating the distance to the origin in a parameter

Accumulating Distance to the Origin

```
def fromOriginAcc(xs: List[Int]) = {
  def inner(xs: List[Int], fromOrigin: Int): List[Int] = {
    xs match {
      case Nil => Nil
      case x :: xs => {
        val xToOrigin = x + fromOrigin
        xToOrigin :: inner(xs, xToOrigin)
      }
    }
  }
  inner(xs, 0)
}
```

Guidelines for Using Accumulators in Functions

- Start with the standard design recipes!
- Add an accumulator *only after* the initial design attempt

Guidelines for Using Accumulators in Functions

- Recognize the benefit of having an accumulator
- Understand what the accumulator denotes

Recognizing the Benefit of an Accumulator

- If the function is structurally recursive and uses an auxiliary function, consider an accumulator
    - Study hand evaluations to see if an accumulator helps in reducing time or space costs

Recognizing the Benefit of an Accumulator

```
def invert[T](xs: List[T]): List[T] = {
  xs match {
    case Nil => Nil
    case x :: xs => makeLastItem(x, invert(xs))
  }
}

def makeLastItem[T](x: T, xs: List[T]): List[T] = {
  xs match {
    case Nil => List(x)
    case y :: ys => y :: makeLastItem(x, ys)
  }
}
```

Recognizing the Benefit of an Accumulator

- there is nothing for invert to forget
- consider accumulating the items walked over

Recognizing the Benefit of an Accumulator

```
def invert[T](xs: List[T]): List[T] = {
  def inner(xs: List[T], accumulator: List[T]): List[T] = {
    xs match {
      case Nil => …
      case y :: ys => … inner(… ys … y … accumulator …)
    }
  }
  inner(xs, Nil)
}
```

Recognizing the Benefit of an Accumulator

- accumulator must stand for a list
- it could stand for all elements that precede `xs`

Recognizing the Benefit of an Accumulator

```
def invert[T](xs: List[T]): List[T] = {
  def inner(xs: List[T], accumulator: List[T]): List[T] = {
    xs match {
      case Nil => …
      case y :: ys => … inner(… ys … y :: accumulator)
    }
  }
  inner(xs, Nil)
}
```

Recognizing the Benefit of an Accumulator

- Now it is clear that the accumulator contains all the elements that precede xs *in reverse order*

Recognizing the Benefit of an Accumulator

```scala
def invert[T](xs: List[T]): List[T] = {
  def inner(xs: List[T], accumulator: List[T]): List[T] = {
    xs match {
      case Nil => accumulator
      case y :: ys => inner(ys, y :: accumulator)
    }
  }
  inner(xs, Nil)
}
```

Recognizing the Benefit of an Accumulator

- The key step in the design process is to establish the invariant that describes the relationship between the accumulator and the parameters of a function

- Establish appropriate accumulator invariant is an art that takes practice

Recognizing the Benefit of an Accumulator

```
def sum1(xs: List[Int]): Int = {
  xs match {
    case Nil => 0
    case y :: ys => y + sum1(ys)
  }
}
```

An Accumulator for Sum

- walking over elements of a list to return their sum
- obvious thing to accumulate is the the sum so far

An Accumulator for Sum

```
def sum2(xs: List[Int]): Int = {
  def inner(xs: List[Int], accumulator: Int): Int = {
    xs match {
      case Nil => accumulator
      case y :: ys => inner(ys, y + accumulator)
    }
  }
  inner(xs, 0)
}
```

An Accumulator for Sum

```
sum1(List(5, 3, 7, 9)) ↦*
5 + sum1(List(3, 7, 9)) ↦*
5 + 3 + sum1(List(7, 9)) ↦*
5 + 3 + 7 + sum1(List(9)) ↦*
5 + 3 + 7 + 9 + sum1(List()) ↦*
5 + 3 + 7 + 9 + 0 ↦
8 + 7 + 9 + 0 ↦
15 + 9 + 0 ↦
24 + 0 ↦
24
```

An Accumulator for Sum

```
sum2(List(5, 3, 7, 9)) ↦*
inner(List(5, 3, 7, 9), 0) ↦*
inner(List(3, 7, 9), 5 + 0) ↦*
inner(List(3, 7, 9), 5) ↦*
inner(List(7, 9), 5 + 3) ↦*
inner(List(7, 9), 8) ↦*
inner(List(9), 7 + 8) ↦*
inner(List(9), 15) ↦*
inner(List(), 9 + 15) ↦*
inner(List(), 24) ↦*
24
```

An Accumulator for Sum

- The key advantage of our accumulator version of sum is space
- The advantage is not a matter as to whether the space is used on the stack or in the heap as an argument!
- The ability to reduce the sum as we recur is the primary cause of space savings

This Would Not Save Space

```
def sum3(xs: List[Int]): Int = {
  def inner(xs: List[Int], accumulator: () => Int): Int = {
    xs match {
      case Nil => accumulator()
      case y :: ys => inner(ys, () => (y + accumulator()))
    }
  }
  inner(xs, () => 0)
}
```

Thoughts on Accumulators

- Accumulator-based functions are not always faster
    - Accumulator-based factorial tends to be slower
- Accumulator-based functions do not always take less space

Thoughts on Accumulators

- Accumulator-based functions are usually harder to understand
- Programmers new to functional programming are seduced by them because sometimes they can be similar to loops

Thoughts on Accumulators

- Use accumulators judiciously and understand the benefits you are trying to achieve

Accumulators and Trees

```
abstract class Tree[+T]

case object Empty extends Tree[Nothing]

case class Branch[+T](data: T, left: Tree[T], right: Tree[T])
extends Tree[T]
```

Accumulators and Trees

```
def height[T](tree: Tree[T]): Int = {
  tree match {
    case Empty => 0
    case Branch(d,l,r) => max(height(l), height(r)) + 1
  }
}
```

Accumulators and Trees

- One natural thing to try is to include an accumulator of type `Int`
- This accumulator can maintain the distance we have descended from the root of the tree

Accumulators and Trees

```scala
def height2[T](tree: Tree[T]): Int = {
  def inner(tree: Tree[T], accumulator: Int): Int = {
    tree match {
      case Empty => accumulator
      case Branch(d,l,r) => max(inner(l, accumulator + 1),
                                inner(r, accumulator + 1))
    }
  }
  inner(tree, 0)
}
```

Family Trees Revisited

```
abstract class FamilyTree

case object Empty extends FamilyTree

case class Cons(father: FamilyTree, mother: FamilyTree,
                name: String, birthYear: Int, eyes: String)
extends FamilyTree
```

Family Trees Revisited

- Let's develop a method **`blueEyedAncestors`** that finds *all* blue-eyed ancestors in a tree

Family Trees Revisited

```scala
def blueEyedAncestors(tree: FamilyTree): List[String] = {
  tree match {
    case Empty => Nil
    case Cons(father,mother,name,_,eyes) => {
      val inParents = blueEyedAncestors(father) ++
                      blueEyedAncestors(mother)

      eyes match {
        case "blue" => name :: inParents
        case _ => inParents
      }
    }
  }
}
```

Family Trees Revisited

- We have defined a structurally recursive function that relies on an auxiliary recursive function: ++
- As discussed, functions of this form often benefit from the use of an accumulator
- We sketch a template for our accumulator-based function in the usual way

Family Trees Revisited

```
def blueEyedAncestors2(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: ...) = {
    tree match {
      case Empty => {...}
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(...father...accumulator...) ...
                        inner(...mother...accumulator...)
        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree...)
}
```

Formulating an Accumulator Invariant

- Our accumulator should remember knowledge about the family tree lost as we descend the tree
- There are two recursive applications: To the father tree and the mother tree
- Options:
    - Denote all blue-eyed ancestors encountered so far
    - Denote all the trees we still need to look at

Option 1: Denote All Blue-Eyed Ancestors Encountered So Far

```
def blueEyedAncestors2(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[String]):
  List[String] = {
    tree match {
      case Empty => accumulator
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, inner(mother, accumulator))

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

Option 1: Denote All Blue-Eyed Ancestors Encountered So Far

```
def blueEyedAncestors2(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[String]):
  List[String] = {
    tree match {
      case Empty => accumulator
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, inner(mother, accumulator))

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

Return type is determined by our choice of accumulator invariant

Option 1: Denote All Blue-Eyed Ancestors Encountered So Far

```
def blueEyedAncestors2(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[String]):
  List[String] = {
    tree match {
      case Empty => accumulator
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, inner(mother, accumulator))

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

We must pass in the result of one descent to the other to maintain the invariant.

Option 1: Denote All Blue-Eyed Ancestors Encountered So Far

```
def blueEyedAncestors2(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[String]):
  List[String] = {
    tree match {
      case Empty => accumulator
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, inner(mother, accumulator))

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

Thus, our combining operator is function composition.

Option 1: Denote All Blue-Eyed Ancestors Encountered So Far

```
def blueEyedAncestors2(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[String]):
  List[String] = {
    tree match {
      case Empty => accumulator
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, inner(mother, accumulator))

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

Our choice of invariant determines what to return in the Empty case.

Option 1: Denote All Blue-Eyed Ancestors Encountered So Far

```
def blueEyedAncestors2(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[String]):
  List[String] = {
    tree match {
      case Empty => accumulator
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, inner(mother, accumulator))

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

Our choice also determines the initial value of the accumulator.

Option 2: Denote All Family Trees Not Yet Processed

```
def blueEyedAncestors3(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[FamilyTree]):
  List[String] = {
    tree match {
      case Empty => {...}
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, mother :: accumulator)

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

We must cons the mother tree on our accumulator for the recursive call to father, to maintain our invariant.

Option 2: Denote All Family Trees Not Yet Processed

```
def blueEyedAncestors3(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[FamilyTree]):
  List[String] = {
    tree match {
      case Empty => {...}
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, mother :: accumulator)

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

Naturally, the only tree to process initially is tree, so our accumulator is Nil.

Option 2: Denote All Family Trees Not Yet Processed

```
def blueEyedAncestors3(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[FamilyTree]):
  List[String] = {
    tree match {
      case Empty => {...}
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, mother :: accumulator)

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

The Empty case is more difficult for this accumulator invariant.

Option 2: Denote All Family Trees Not Yet Processed

- When the tree is empty, we choose the next element in our accumulator to recur on

Option 2: Denote All Family Trees Not Yet Processed

```scala
def blueEyedAncestors3(tree: FamilyTree): List[String] = {
  def inner(tree: FamilyTree, accumulator: List[FamilyTree]): List[String] = {
    tree match {
      case Empty => accumulator match {
        case Nil => Nil
        case tree :: trees => inner(tree, trees)
      }
      case Cons(father,mother,name,_,eyes) => {
        val inParents = inner(father, mother :: accumulator)

        eyes match {
          case "blue" => name :: inParents
          case _ => inParents
        }
      }
    }
  }
  inner(tree, Nil)
}
```

Tail Recursion

Tail Recursion

- Some functions defined using accumulators have a special property:
    - The recursive call occurs as the last step in the computation

Nats

```
abstract class Nat {
  def !(): Nat
  def *(m: Nat): Nat
  def +(m: Nat): Nat
}
```

*Note that this is a postfix operator.
(This follows from the rules for
method application syntax.)*

Nats

```
case object Zero extends Nat {
  def !() = Next(Zero)
  def *(m: Nat) = Zero
  def +(m: Nat) = m
}
```

Nats

```
case class Next(n: Nat) extends Nat {
  def !() = this * (n!)
  def *(m: Nat) = m + (n * m)
  def +(m: Nat) = Next(n + m)
}
```

Nats

```
Next(Next(Next(Zero)))! ↦
Next(Next(Next(Zero))) * Next(Next(Zero))! ↦
Next(Next(Next(Zero))) * Next(Next(Zero)) * Next(Zero)! ↦
Next(Next(Next(Zero))) * Next(Next(Zero)) * Next(Zero) * Zero! ↦
Next(Next(Next(Zero))) * Next(Next(Zero)) * Next(Zero) * Next(Zero) ↦
…
Next(Next(Next(Next(Next(Next(Zero))))))
```

Pure Recursion

```
def !() = this * (n!)
```

Tail Recursion

```
 def !() = {
   def inner(n: Nat, acc: Nat): Nat = {
     n match {
       case Zero => acc
       case Next(m) => inner(m, n * acc)
     }
   }
   inner(this, Next(Zero))
 }
}
```

Nats

```
Next(Next(Next(Zero)))! ↦
inner(Next(Next(Next(Zero))), Next(Zero)) ↦
inner(Next(Next(Zero)), Next(Next(Next(Zero)))) ↦
inner(Next(Zero), Next(Next(Next(Next(Next(Next(Zero))))))) ↦
inner(Zero, Next(Next(Next(Next(Next(Next(Zero))))))) ↦
Next(Next(Next(Next(Next(Next(Zero))))))
```

Translating for Ints

```
def factorial(n: Int): Int = {
  if (n == 0) 1
  else n * factorial(n - 1)
}

def factorial2(n: Int) = {
  def inner(n: Int, acc: Int): Int = {
    if (n == 0) acc
    else inner(n - 1, n * acc)
  }
  inner(n, 1)
}
```

Pure Recursion with Ints

```
3! ↦
3 * 2! ↦
3 * 2 * 1! ↦
3 * 2 * 1 * 0! ↦
3 * 2 * 1 * 1 ↦
…
6
```

Tail Recursion with Ints

```
3! ↦
inner(3, 1) ↦
inner(2, 3) ↦
inner(1, 6) ↦
inner(0, 6) ↦
6
```