Comp 311
Functional Programming

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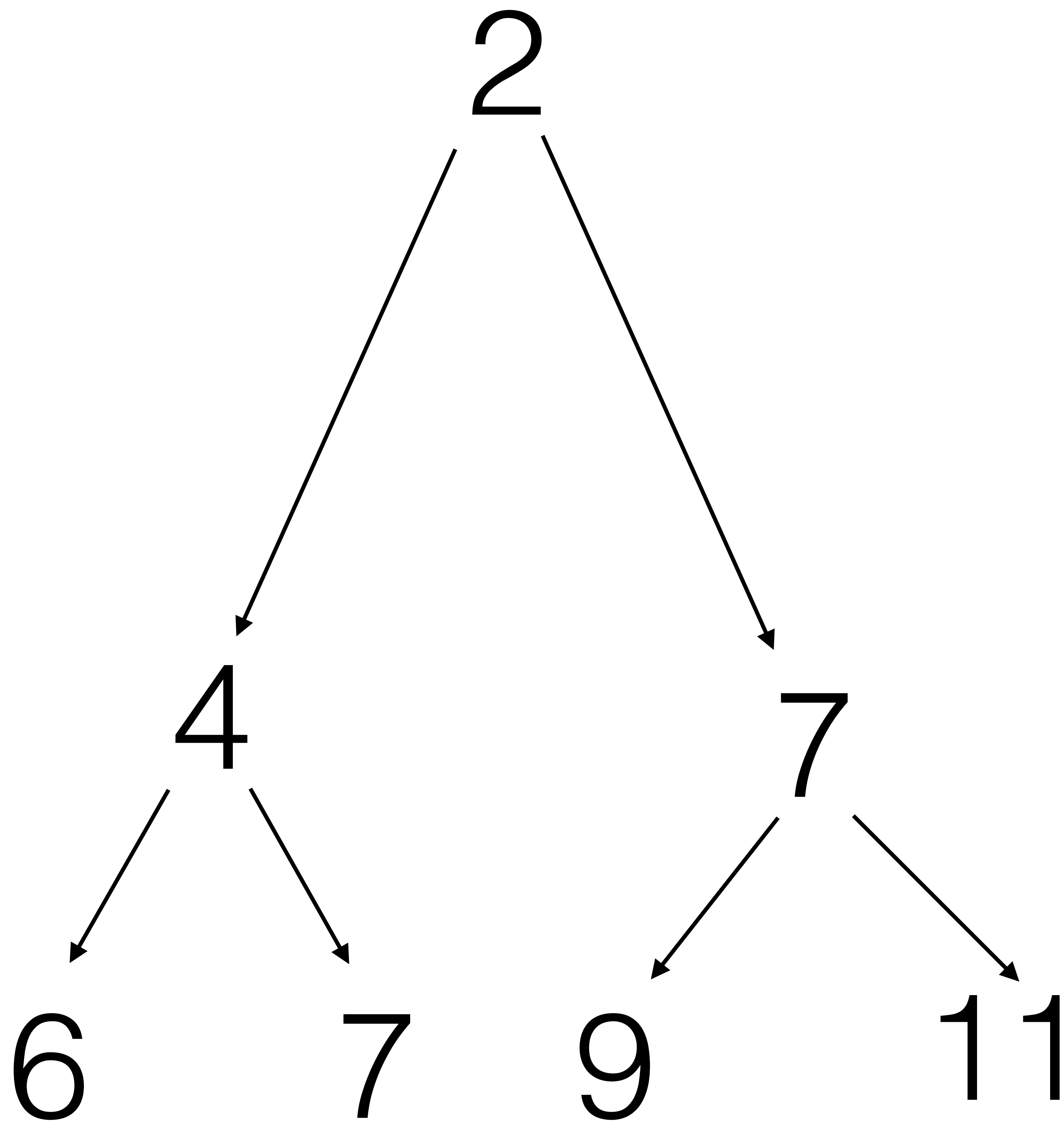
Functional Data Structures

Leftist Heaps

Leftist Heaps

- Often in a collection of elements we only need to access the *minimum* element
- A data structure that supports access only to the minimum element is called a *heap:*
    - A tree in which the element at the root of each subtree is the minimum element of that subtree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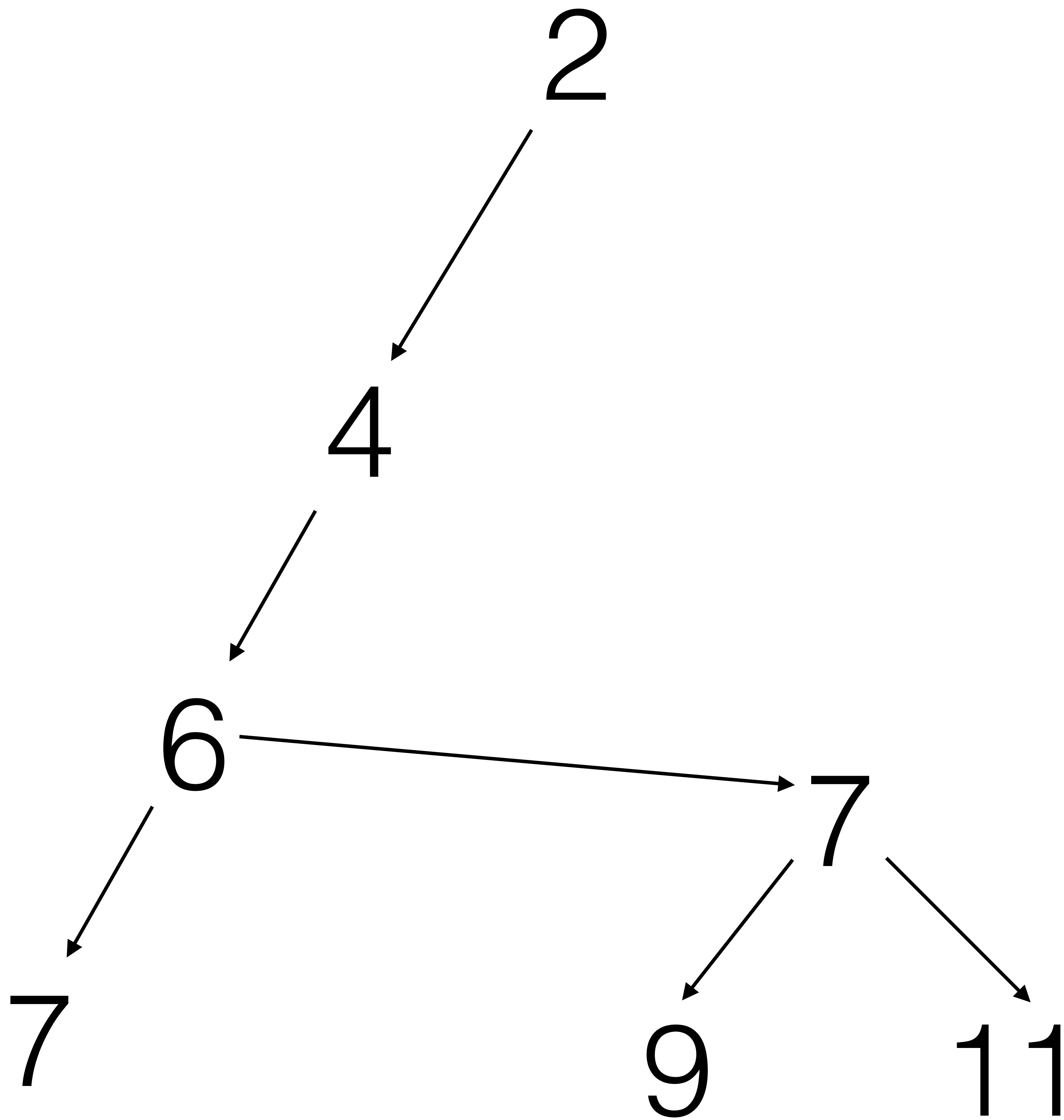
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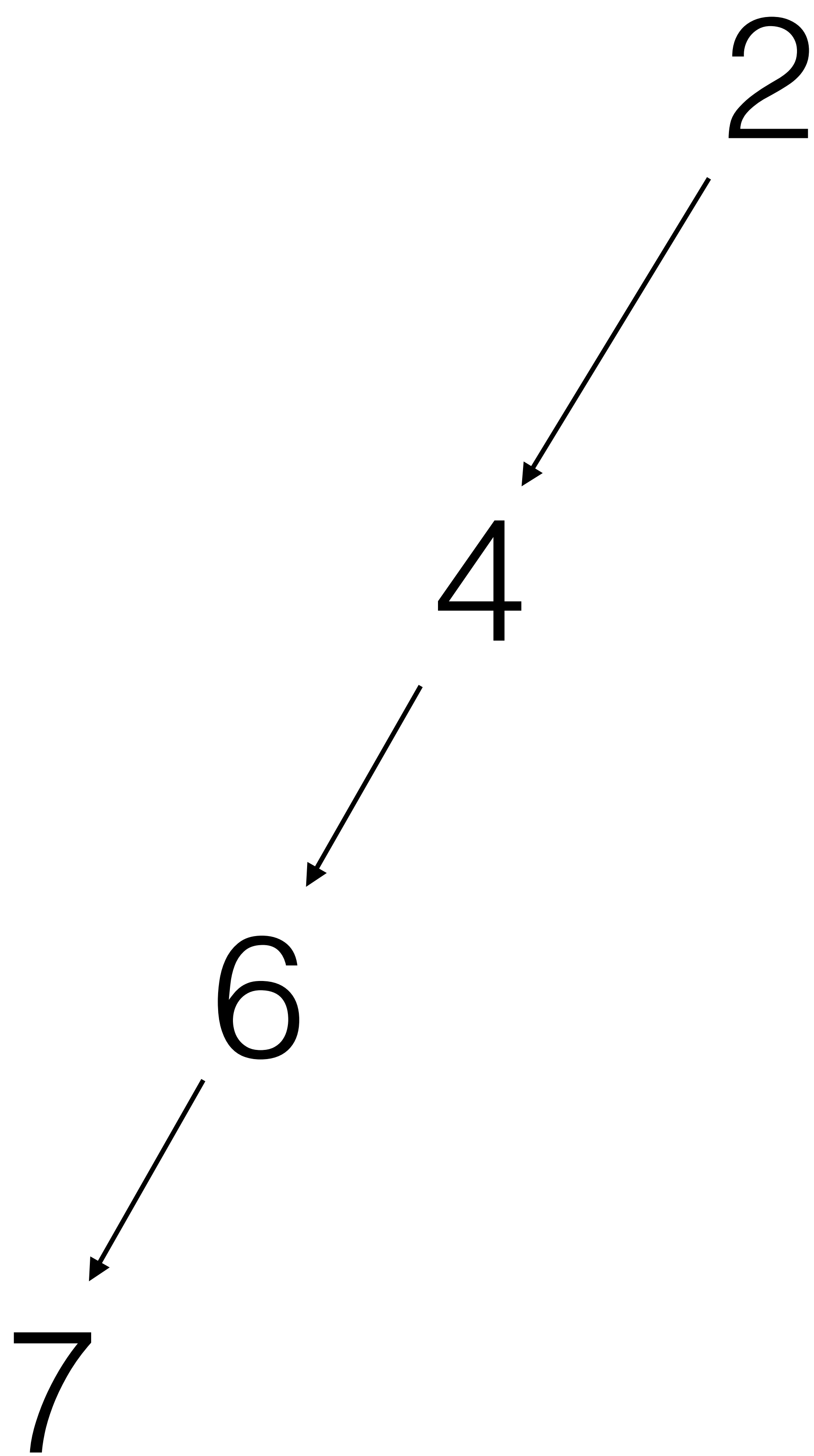
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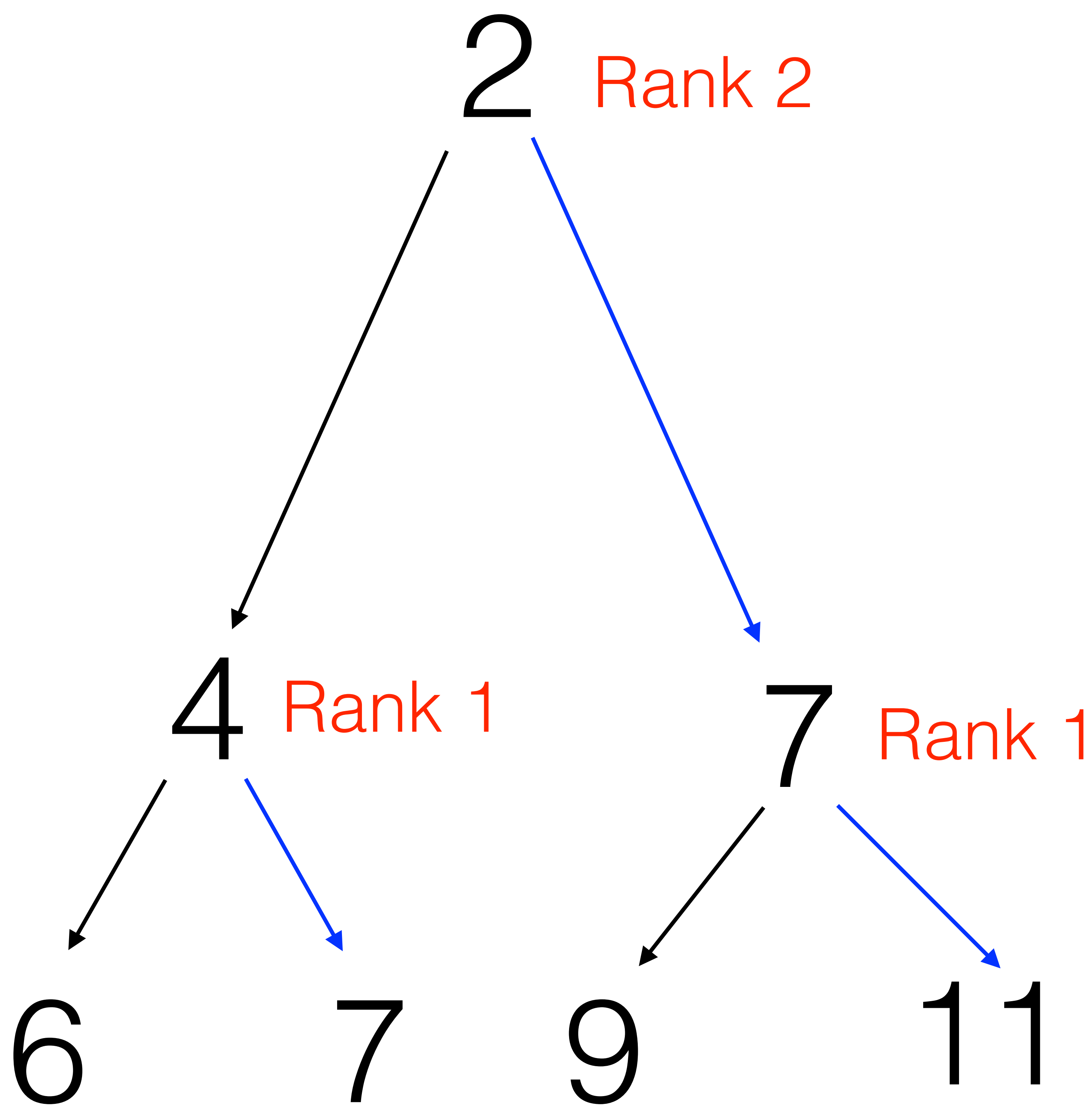
Leftist Heaps

- Let the *rank* of a node be the length of its *right spine*
- Then a *leftist heap* also satisfies the following property:
    - The rank of a left child is no smaller than the rank of its sibling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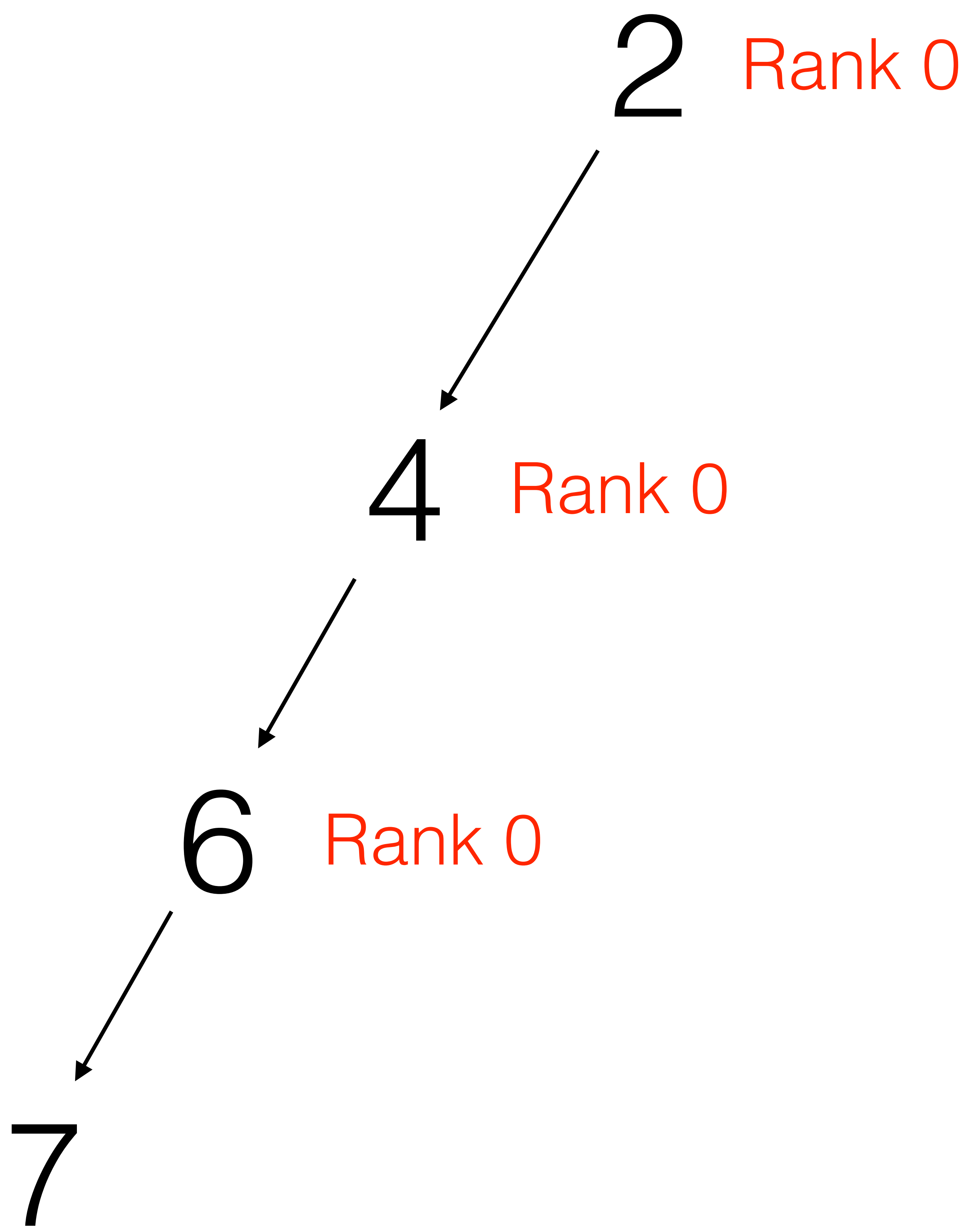
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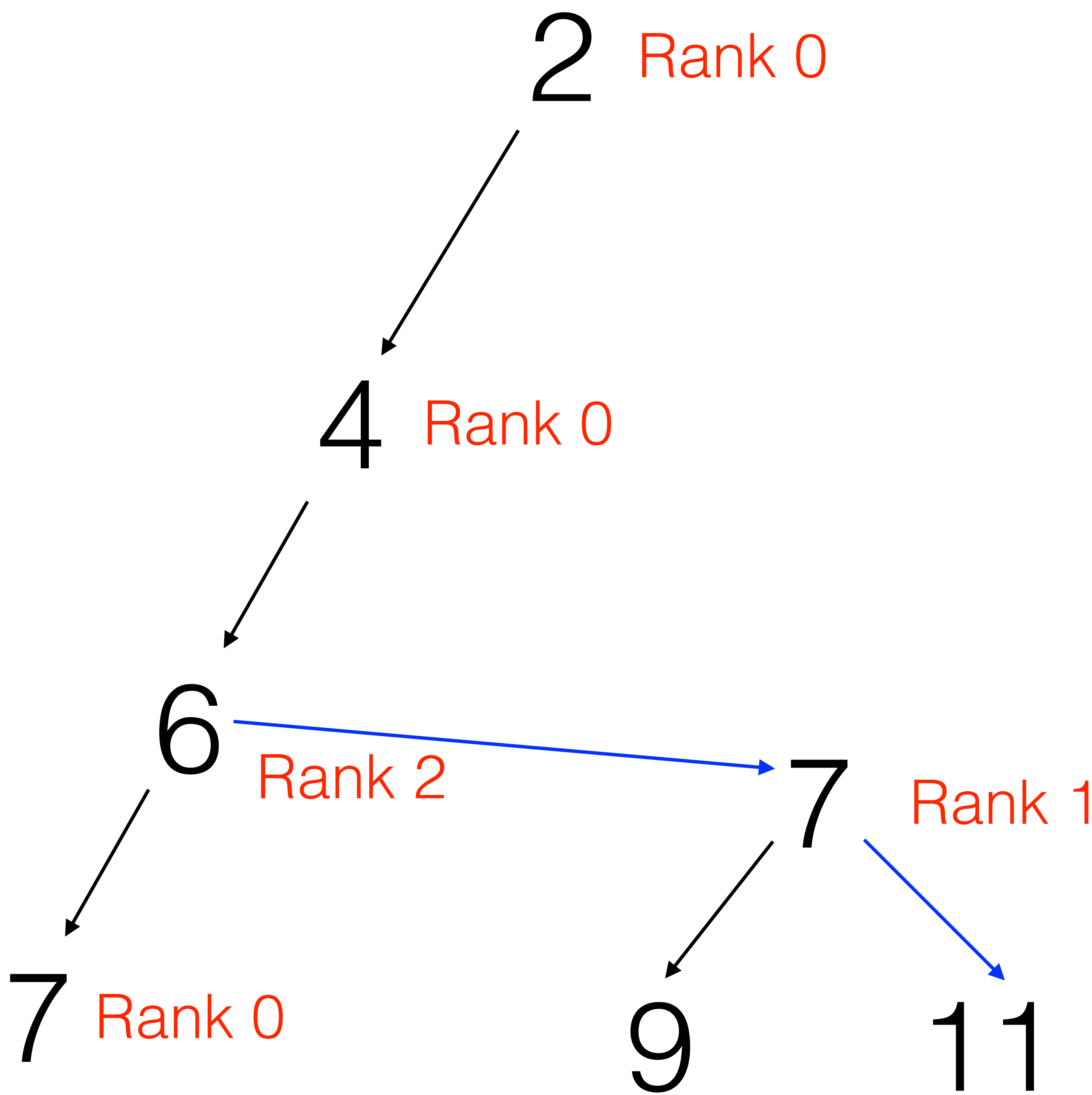
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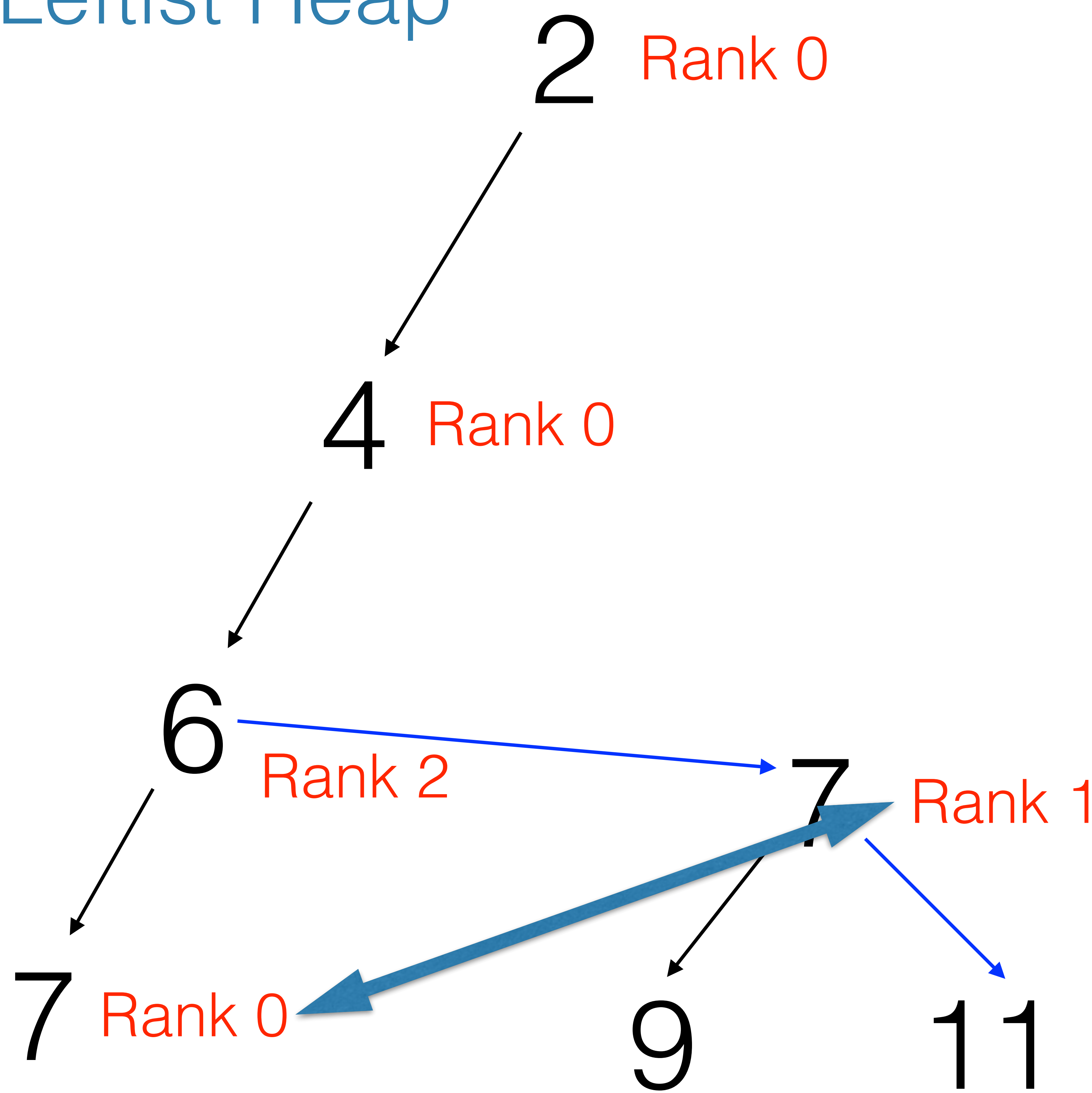
Not a Leftist Heap



Consequences of the Leftist Property

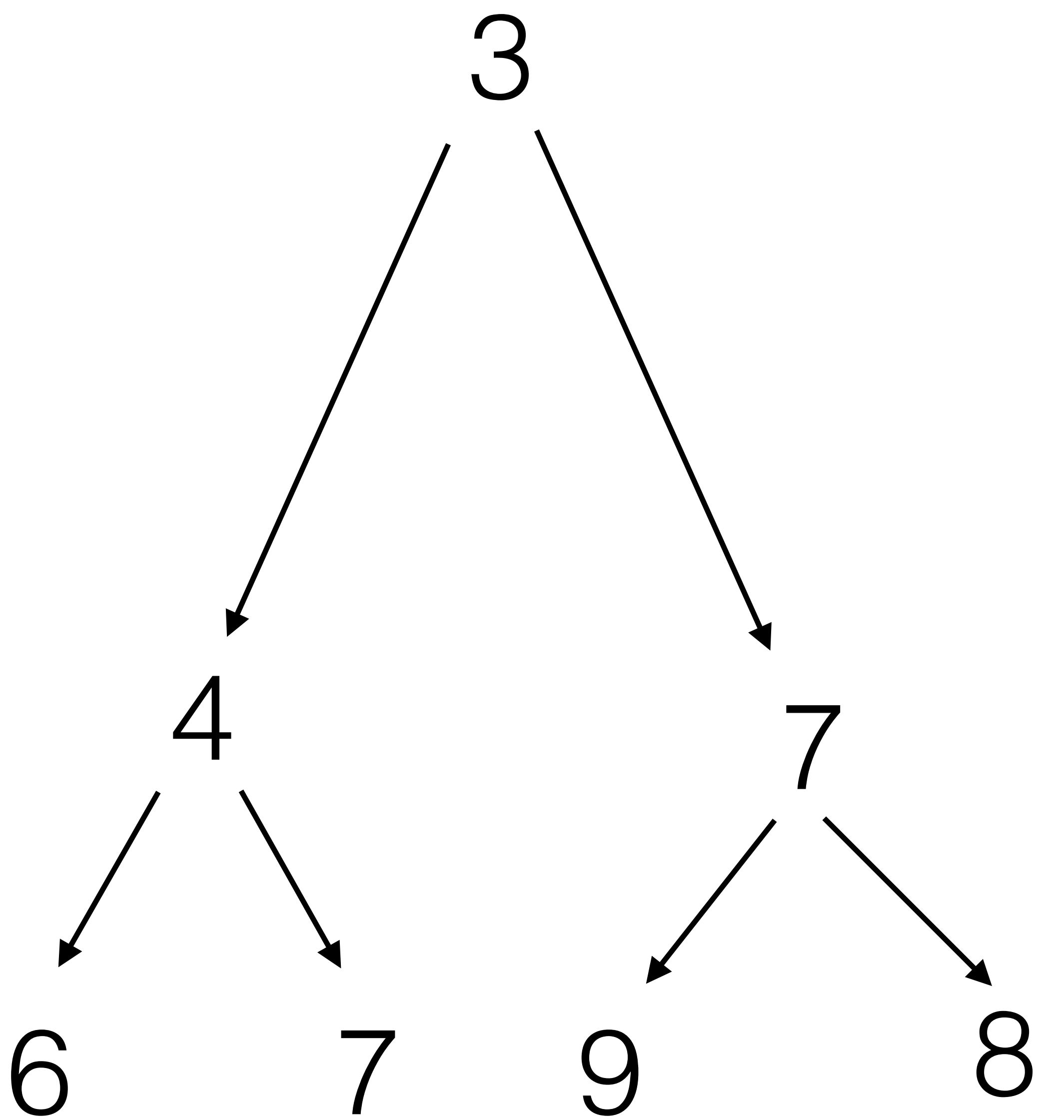
- The right spine of a node is always the shortest path to a leaf

- The right spine of a node contains O(log n) elements in the worst case

- The elements along the right spines are in sorted order

Efficient Merging of Two Leftist Heaps

- Intuitively, we can merge two leftist heaps by:
  - Merging their right spines as if they were sorted lists
  - Swapping child nodes along the merged right spine as needed to preserve the leftist property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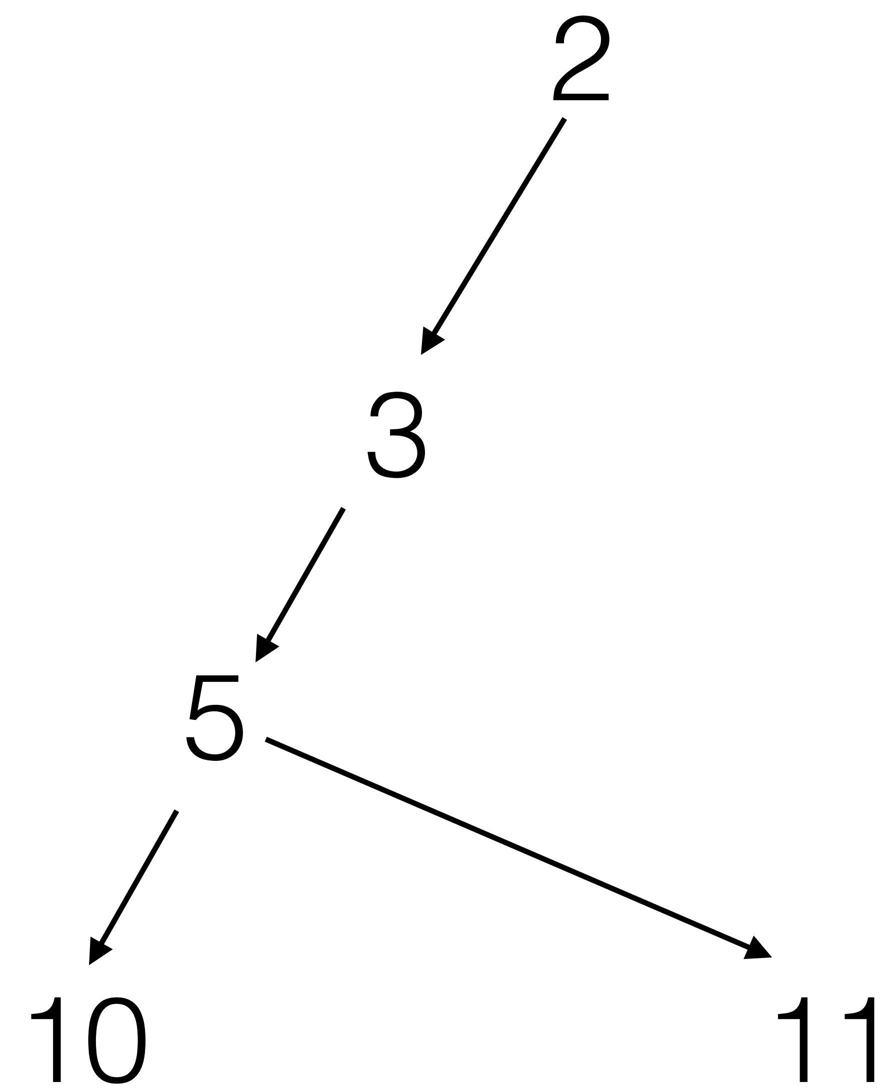
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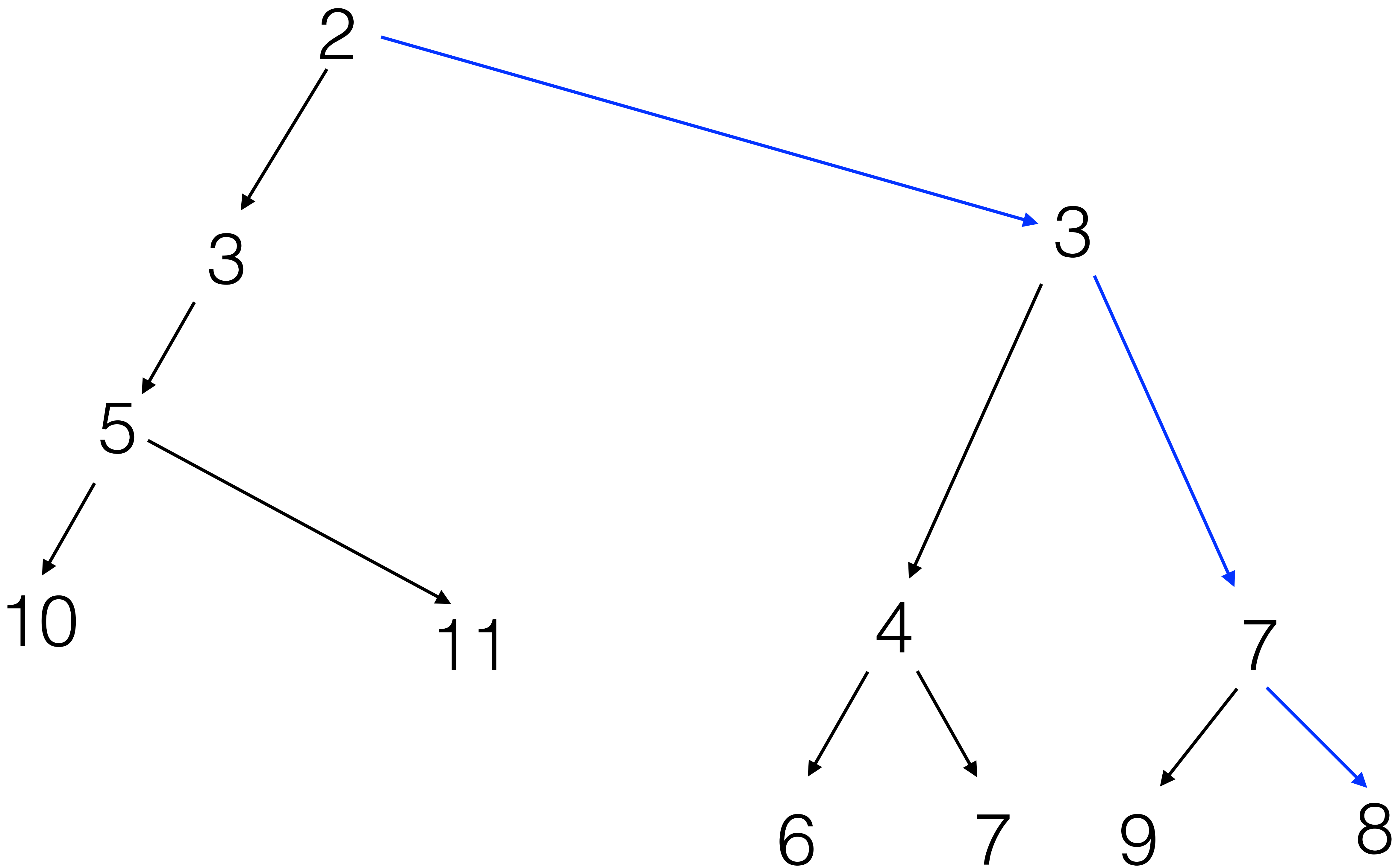
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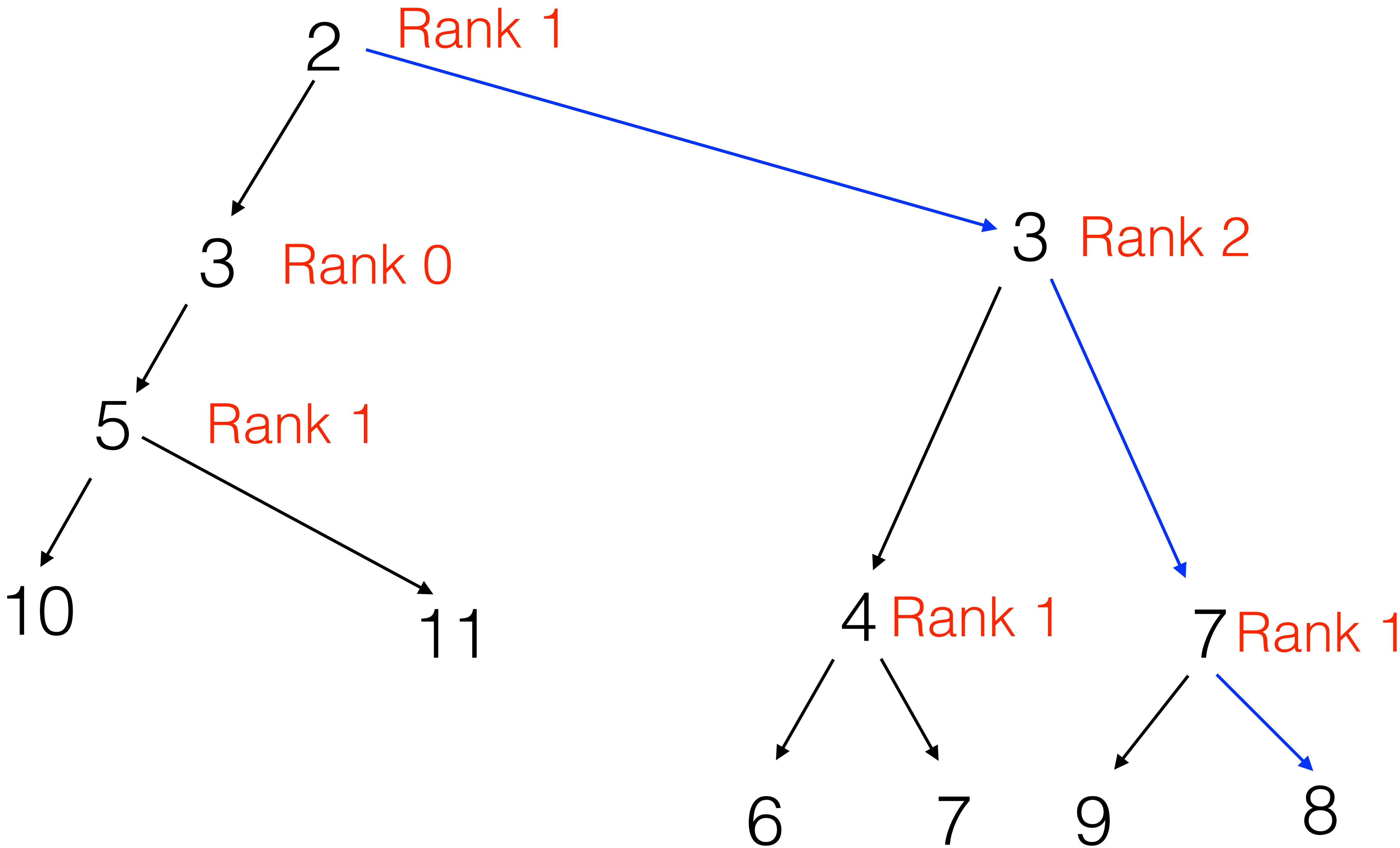
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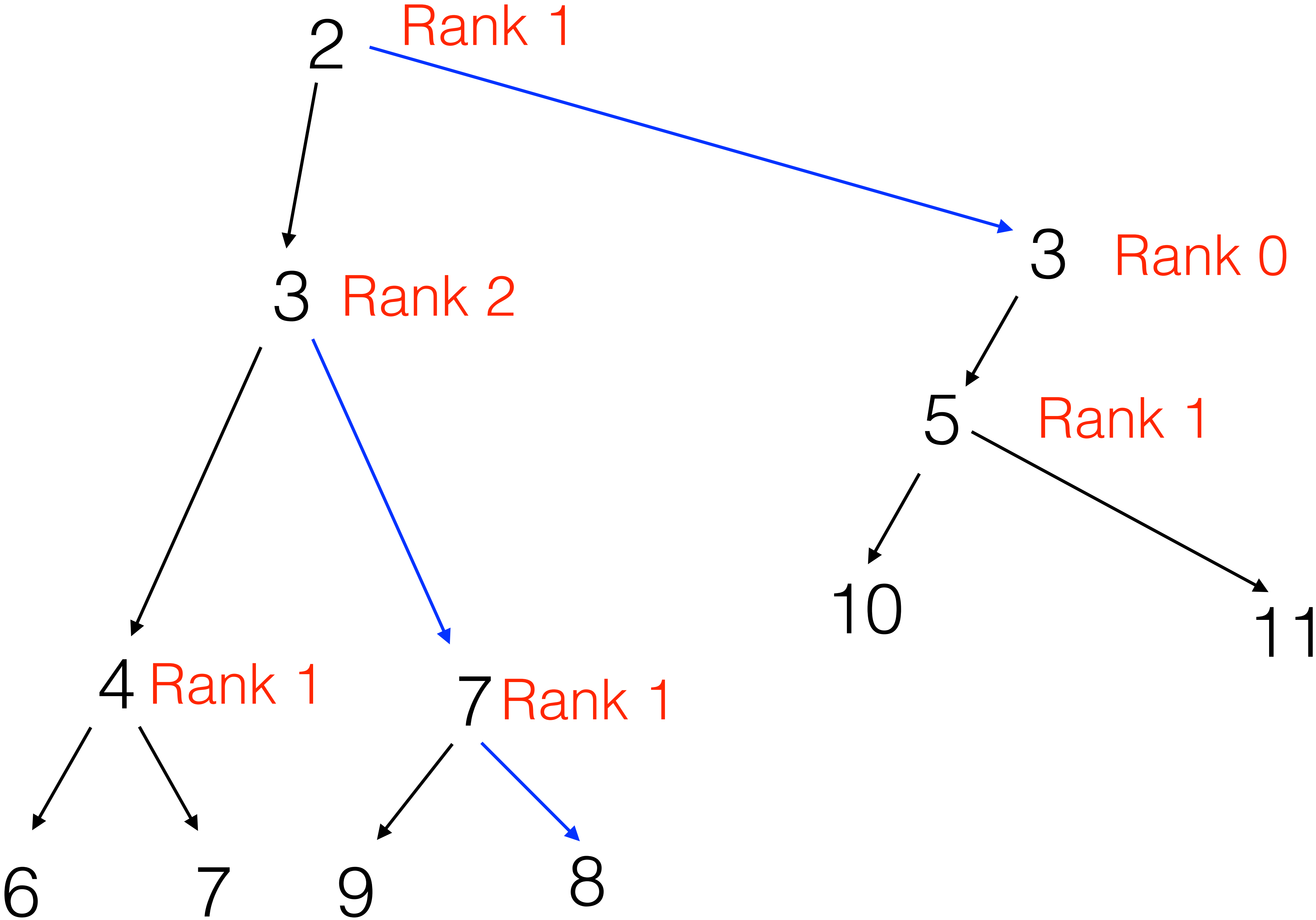

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Rank 1
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Rank 2
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Rank 0
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Rank 1
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Rank 1
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Rank 1
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Leftist Heaps

```scala
case class Leaf[T <: Ordered[T]]() extends Heap[T] {
  def rank = 0
  def isEmpty = true

  def merge(that: Heap[T]) = that

  def min = throw new Error("Attempt to call min on an empty heap")
  def deleteMin = throw new Error("Attempt to call deleteMin on an empty heap")
}
```

Leftist Heaps

```
case class Branch[T <: Ordered[T]]
  (rank: Int, x: T, left: Heap[T], right: Heap[T]) extends Heap[T] {

  def isEmpty = false

  def merge(that: Heap[T]) = {
    that match {
      case Leaf() => this
      case Branch(_, y, l, r) =>
        if (x <= y) makeBranch(x, left, right merge that)
        else makeBranch(y, l, this merge r)
    }
  }
  def min = x
  def deleteMin = left merge right
}
```

Leftist Heaps

```
abstract class Heap[T <: Ordered[T]] {
  def empty = Leaf[T]
  def isEmpty: Boolean

  def insert(element: T): Heap[T] =
    this merge Branch(1, element, empty, empty)

  def merge(that: Heap[T]): Heap[T]

  /* require (! isEmpty) */
  def min: T

  /* require (! isEmpty) */
  def deleteMin: Heap[T]

  def rank: Int

  def makeBranch(x: T, a: Heap[T], b: Heap[T]) = {
    if (a.rank >= b.rank) Branch(b.rank + 1, x, a, b)
    else Branch(a.rank + 1, x, b, a)
  }
}
```

Red-Black Trees

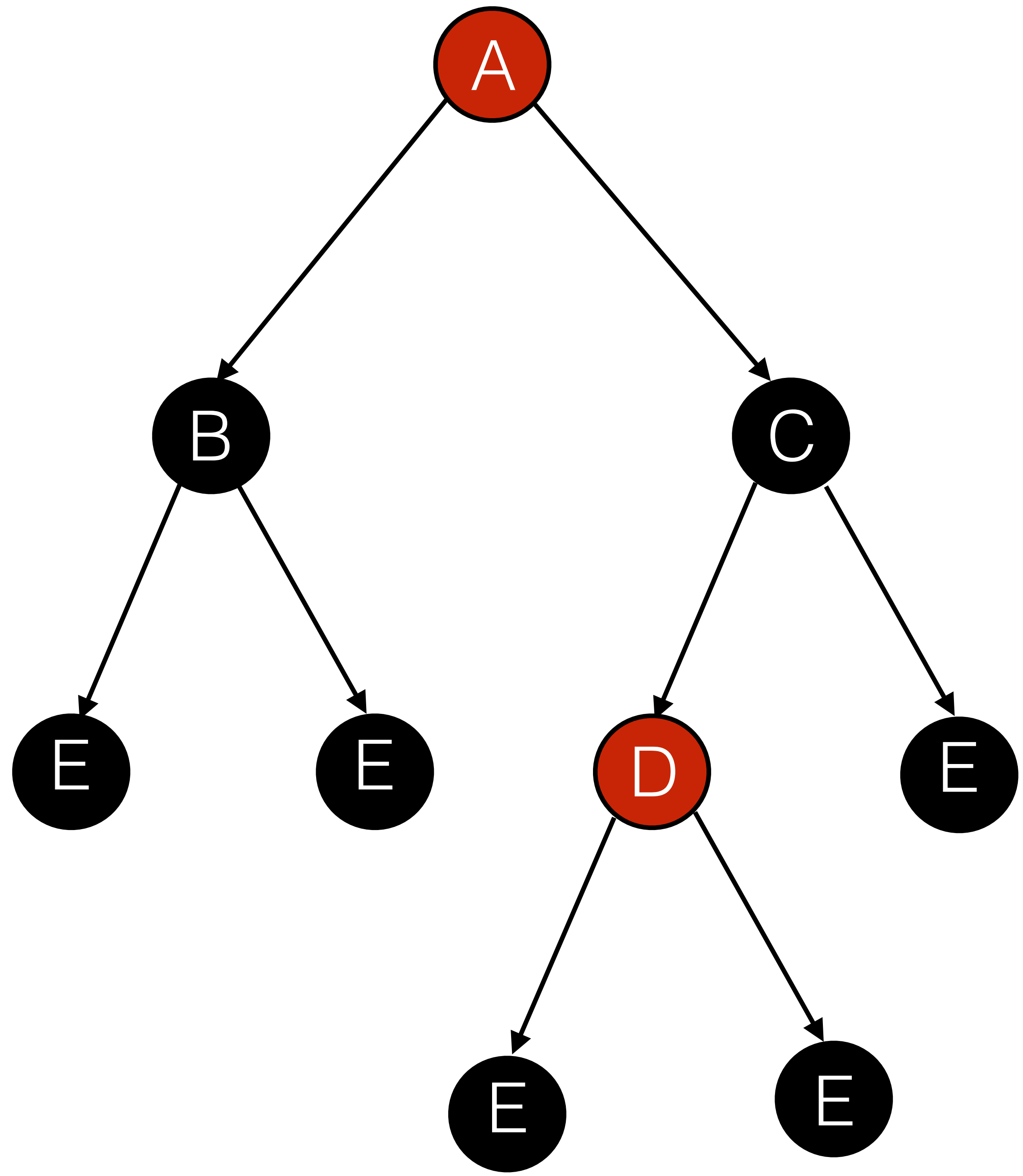
Red-Black Trees

- With naive binary search trees, lookup can take O(n) time in the worst case
- We can fix this problem by rebalancing the trees as we add elements
- Red-Black trees are one approach to keeping the trees approximately balanced

Red-Black Trees

- Every node is colored either red or black
- All leaf nodes are black
- No red node has a red child
- Every path from the root to a leaf contains the same number of black nodes

An Example Red-Black Tree



Red-Black Trees

- These invariants imply that:
  - The longest possible path from the root to a leaf consists of an alternating sequence of red nodes and black nodes
  - The shortest possible path from the root to a leaf consists of all black nodes
- Thus, there is at most a factor of two difference in length between the shortest and longest paths

Red-Black Trees

```
sealed abstract class Color
case object Red extends Color
case object Black extends Color
```

Red-Black Trees

```
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case object Red extends Color
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```

All subclasses of a sealed class must be defined in the same file as the sealed class.

Red-Black Trees

```
sealed abstract class Color
case object Red extends Color
case object Black extends Color
```

Pattern matching against a sealed class is checked to ensure exhaustiveness.

Strategy for Insertion

- We insert new elements as usual, but then rebalance the tree to maintain the red-black invariants
- At the end of the rebalancing, we recolor the root to black
  - This last step cannot violate our invariants

Red-Black Trees

```
abstract class Tree[T <: Ordered[T]] {
  def empty = Leaf[T]
  def contains(x: T): Boolean
  def insert(x: T): Tree[T] = insertChildren(x) match {
    case Branch(c,l,e,r) => Branch(Black, l, e, r)
  }
  def insertChildren(x: T): Branch[T]
}
```

We call a helper function insertChildren, which performs the insertion and rebalancing.

Red-Black Trees

```
abstract class Tree[T <: Ordered[T]] {
  def empty = Leaf[T]
  def contains(x: T): Boolean
  def insert(x: T): Tree[T] = insertChildren(x) match {
    case Branch(c,l,e,r) => Branch(Black, l, e, r)
  }
  def insertChildren(x: T): Branch[T]
}
```

We take the result from insertChildren, ignore the color of the root and return a tree that is nearly identical except that the root is colored black.

Red-Black Trees

```scala
case class Leaf[T <: Ordered[T]]() extends Tree[T] {
  def contains(x: T) = false
  def insertChildren(x: T) = Branch(Red, this, x, this)
}
```

Red-Black Trees

```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {

  def contains(x: T) = {
    if (x < element) left contains x
    else if (x > element) right contains x
    else true // x == element
  }
  …
}
```

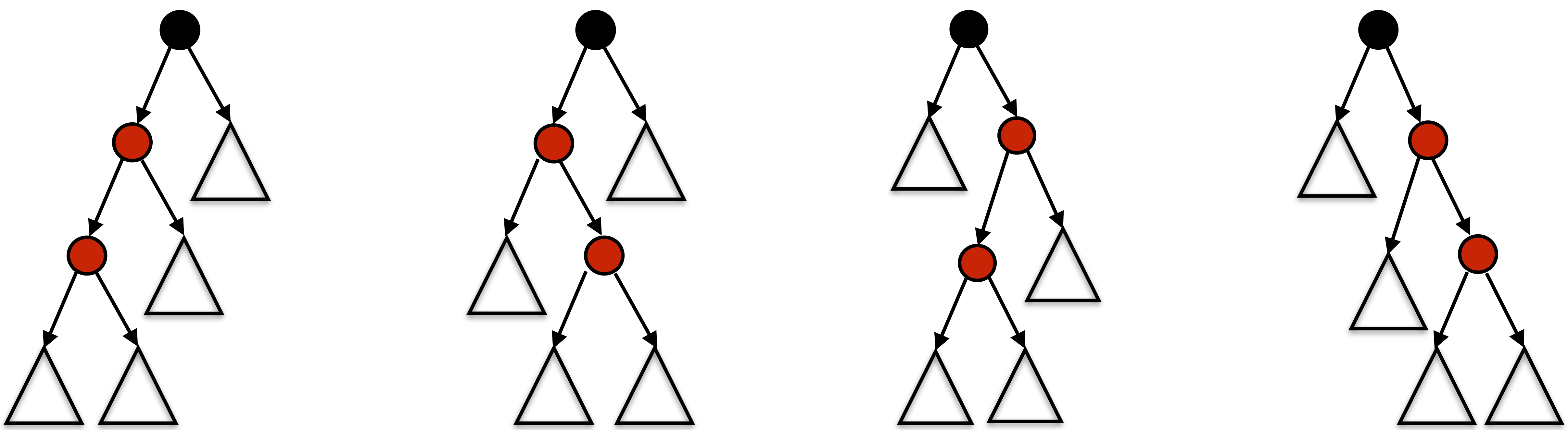
Red-Black Trees

```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {
  …
  def insertChildren(x: T) = {
    if (x < element)
      balance(color, left insertChildren x, element, right)
    else if (x > element)
      balance(color, left, element, right insertChildren x)
    else this
  }
  …
}
```

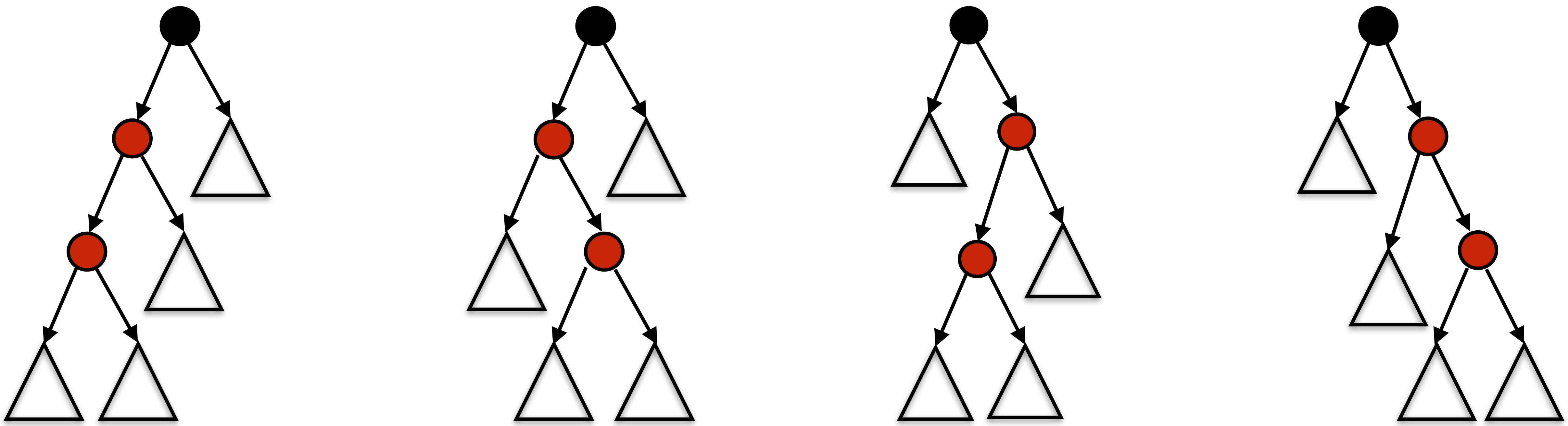
Rebalancing

- Because the base case of insertChildren (at a leaf node) always inserts a red node, the number of black nodes along each path is unaffected
- However, the new tree might contain a red node with a red child

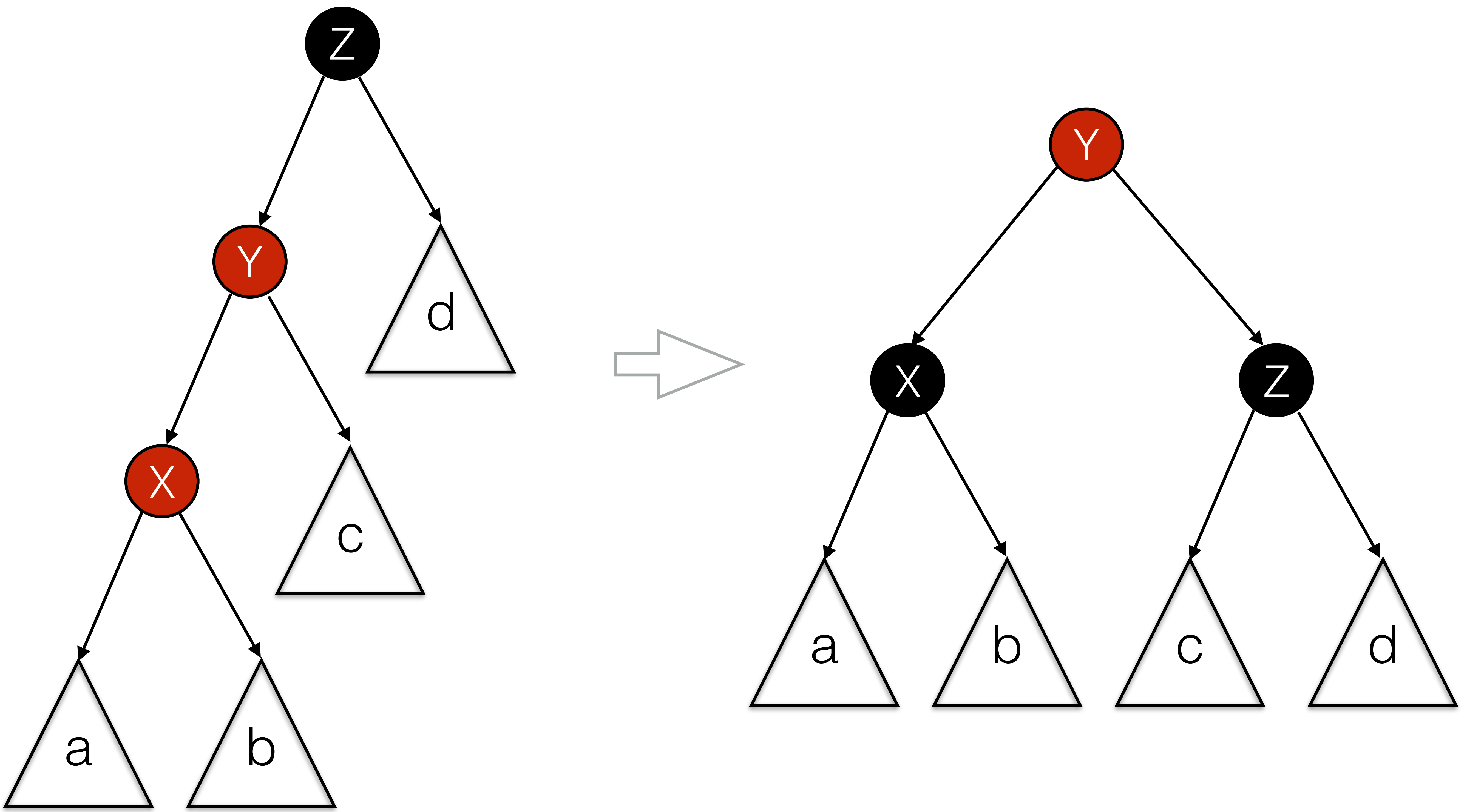
Rebalancing:
There are Four Cases to Consider



Rebalancing:
There are Four Cases to Consider



We use pattern matching to enumerate the cases.


Z
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d

```scala
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {










      …
    }
  }
  …
```

```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
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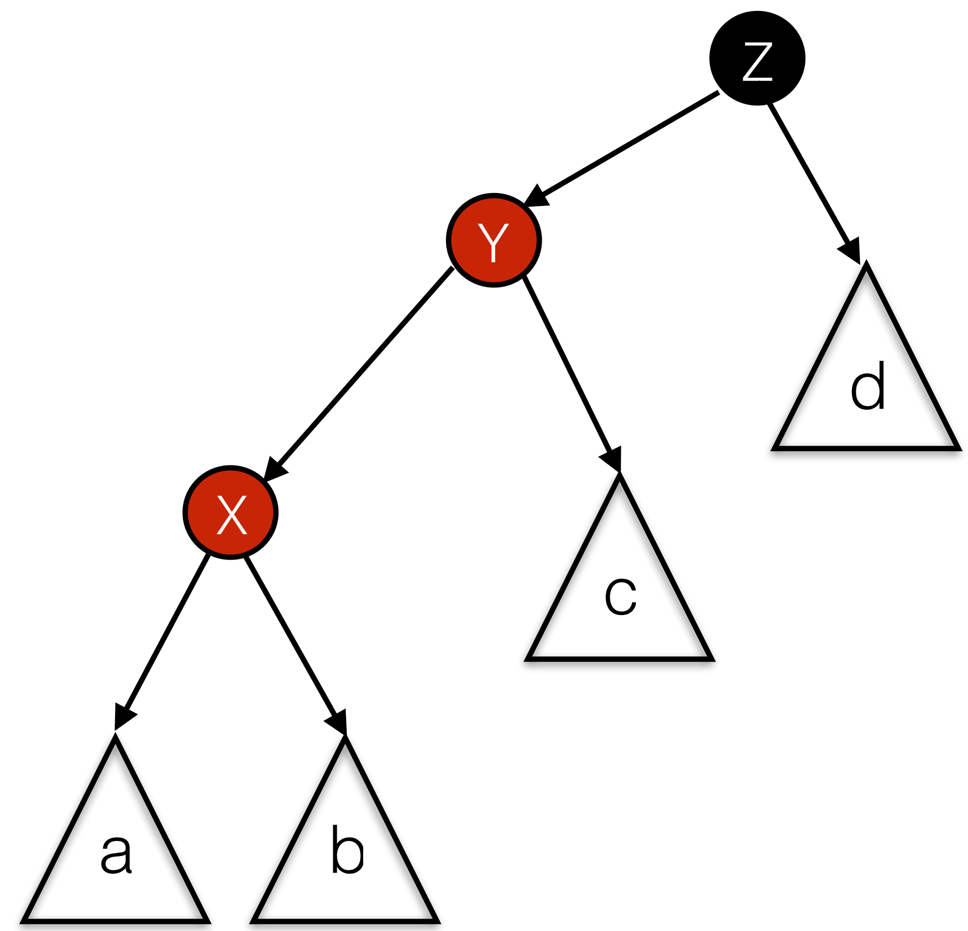


```
      ...
    }
  }
  ...
```

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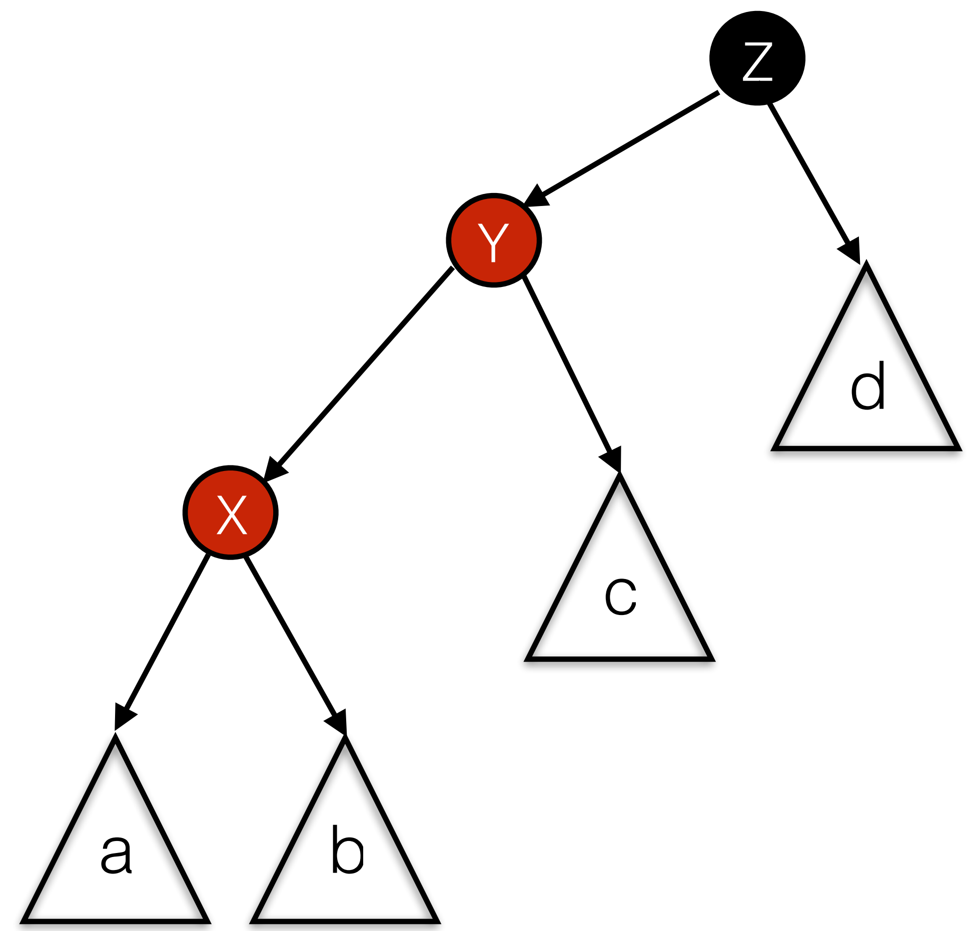
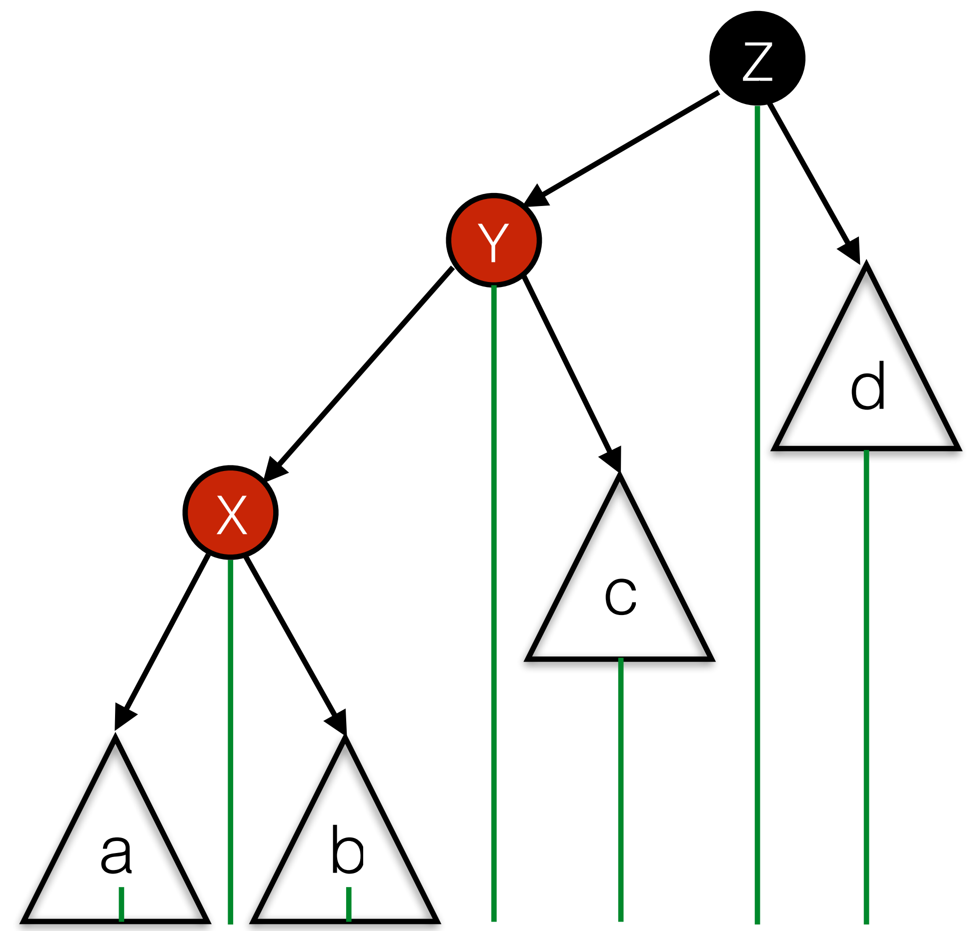
```




```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>

      …
    }
  }
…
```

```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
```




```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>

      …
    }
  }
  …
```

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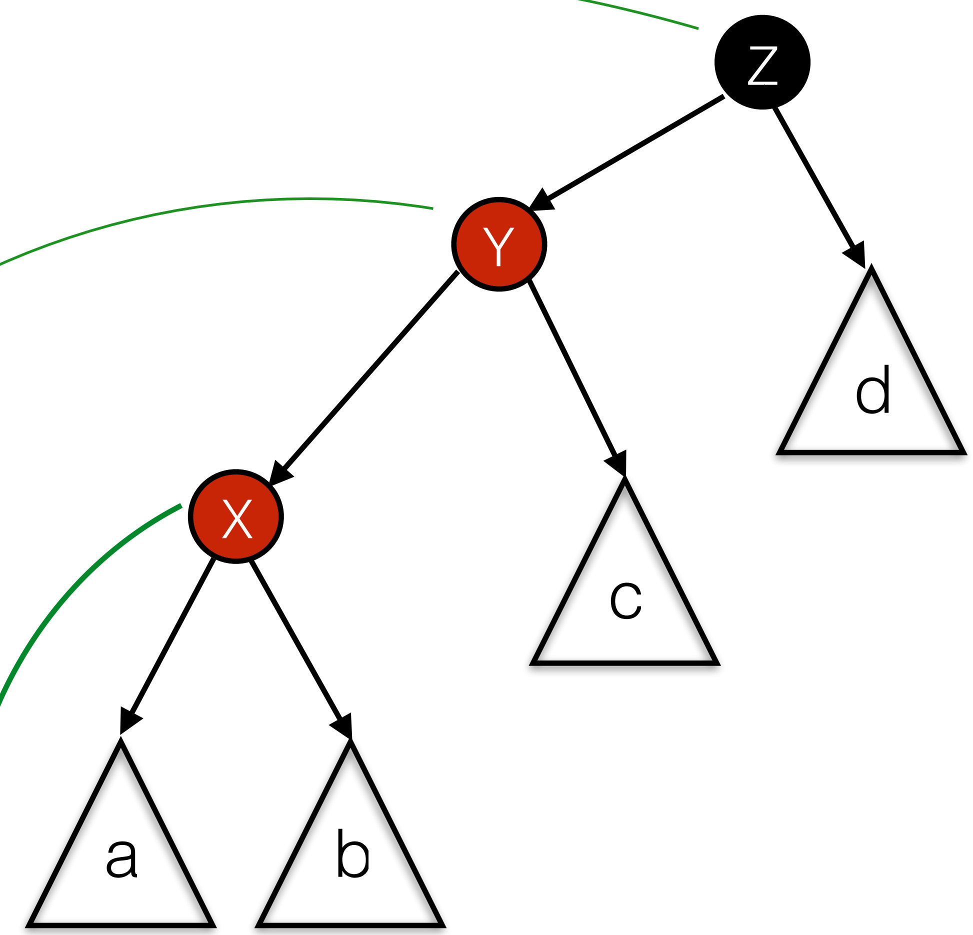


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      …
    }
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  …
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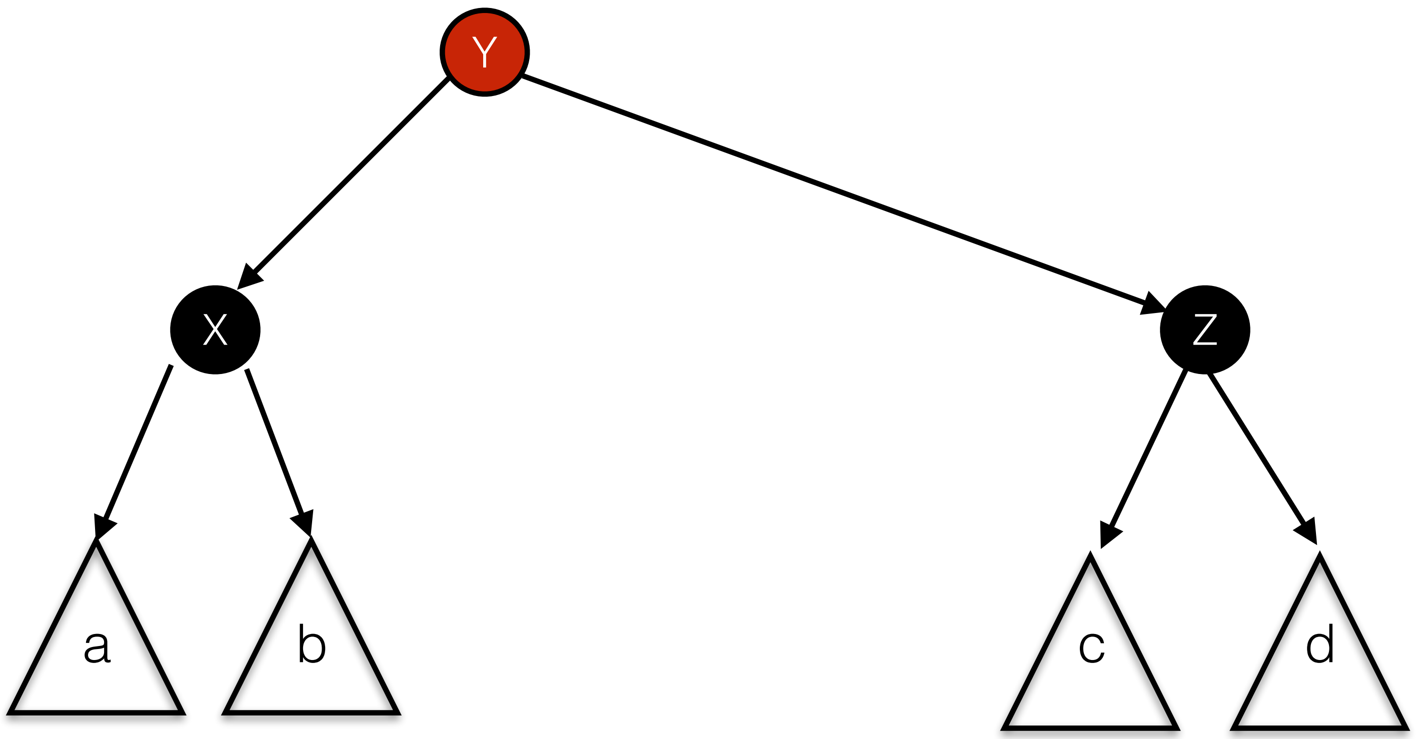


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      …
    }
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  …
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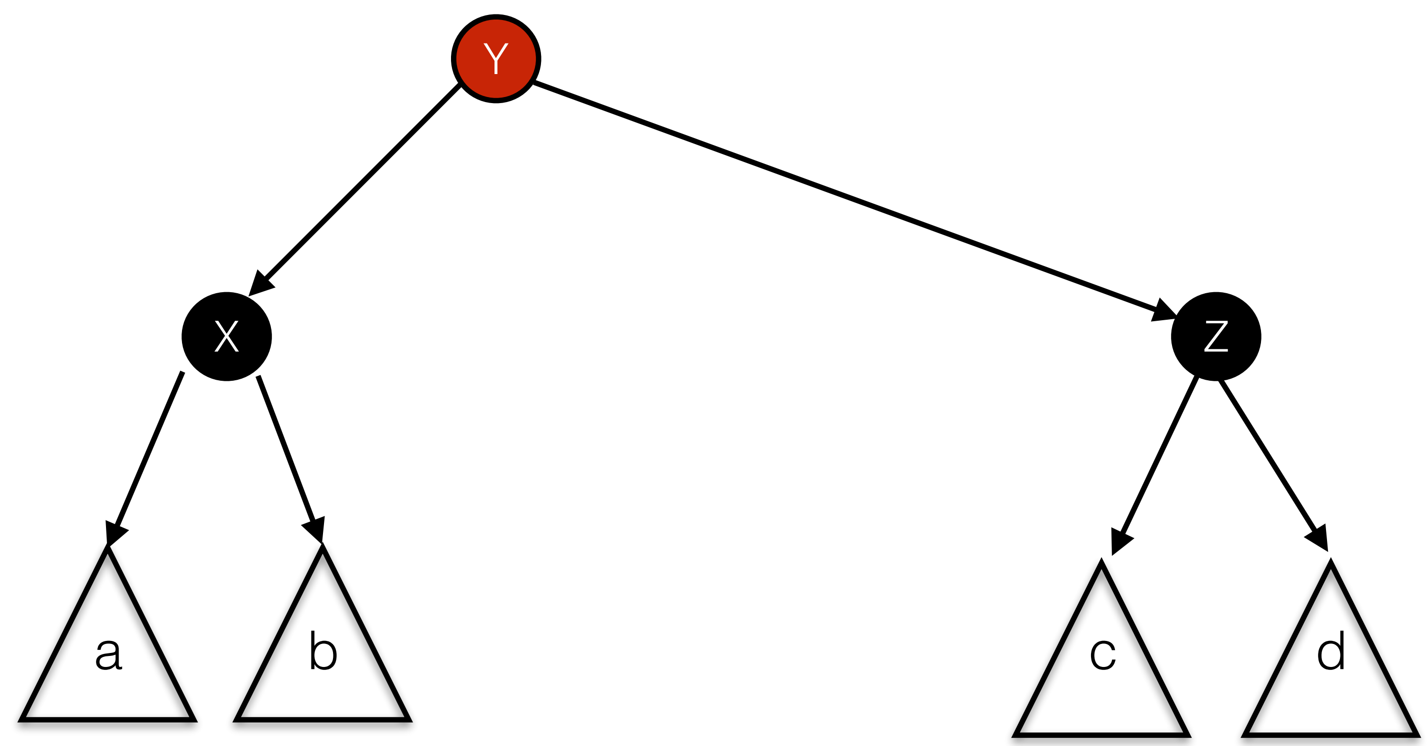


```
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    …
  }
}
…
```

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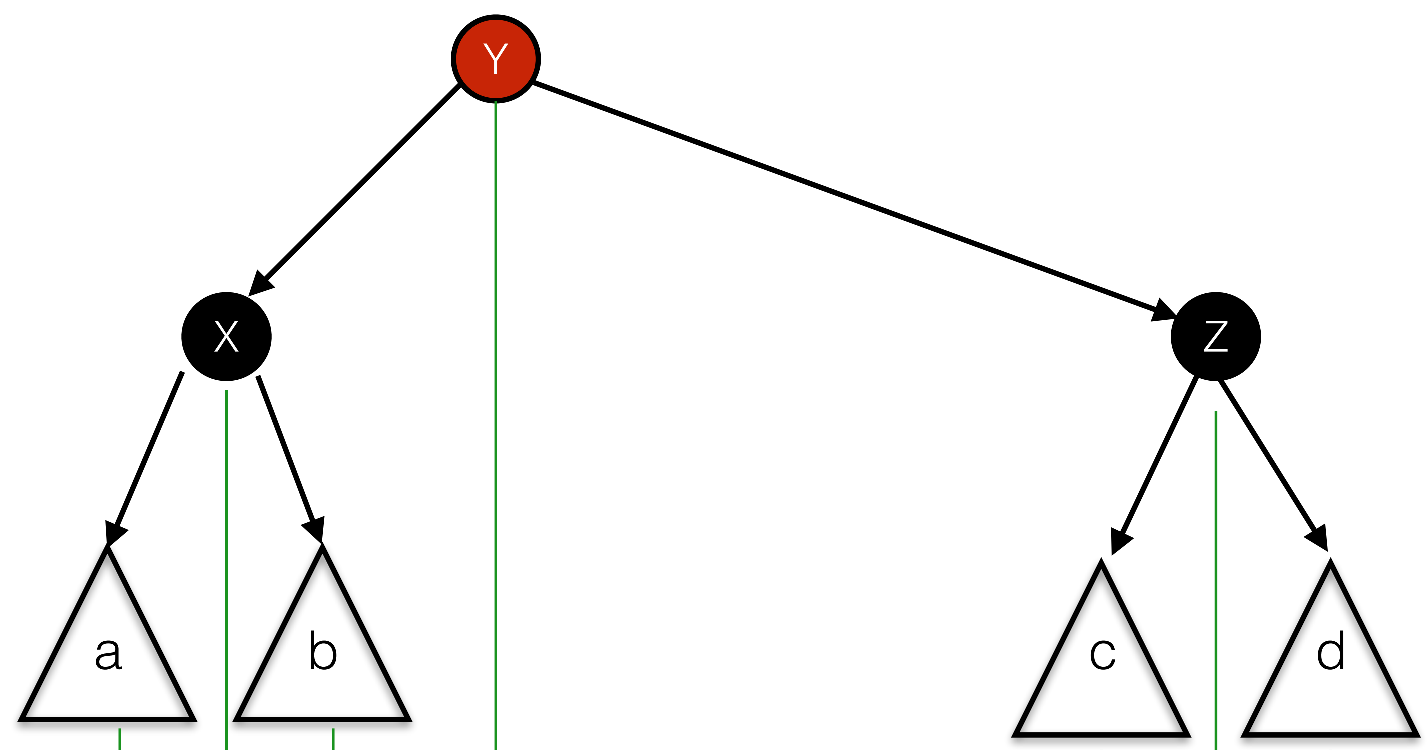


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      …
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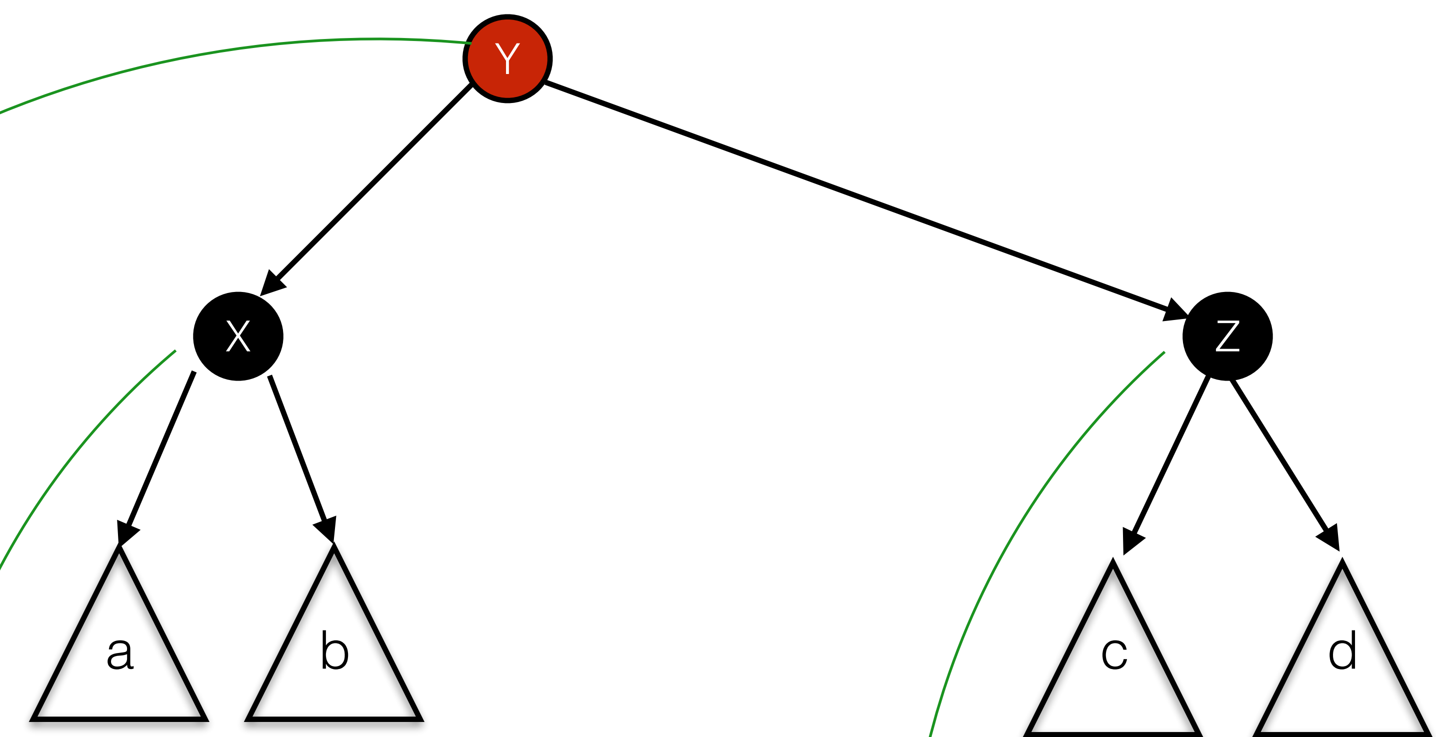


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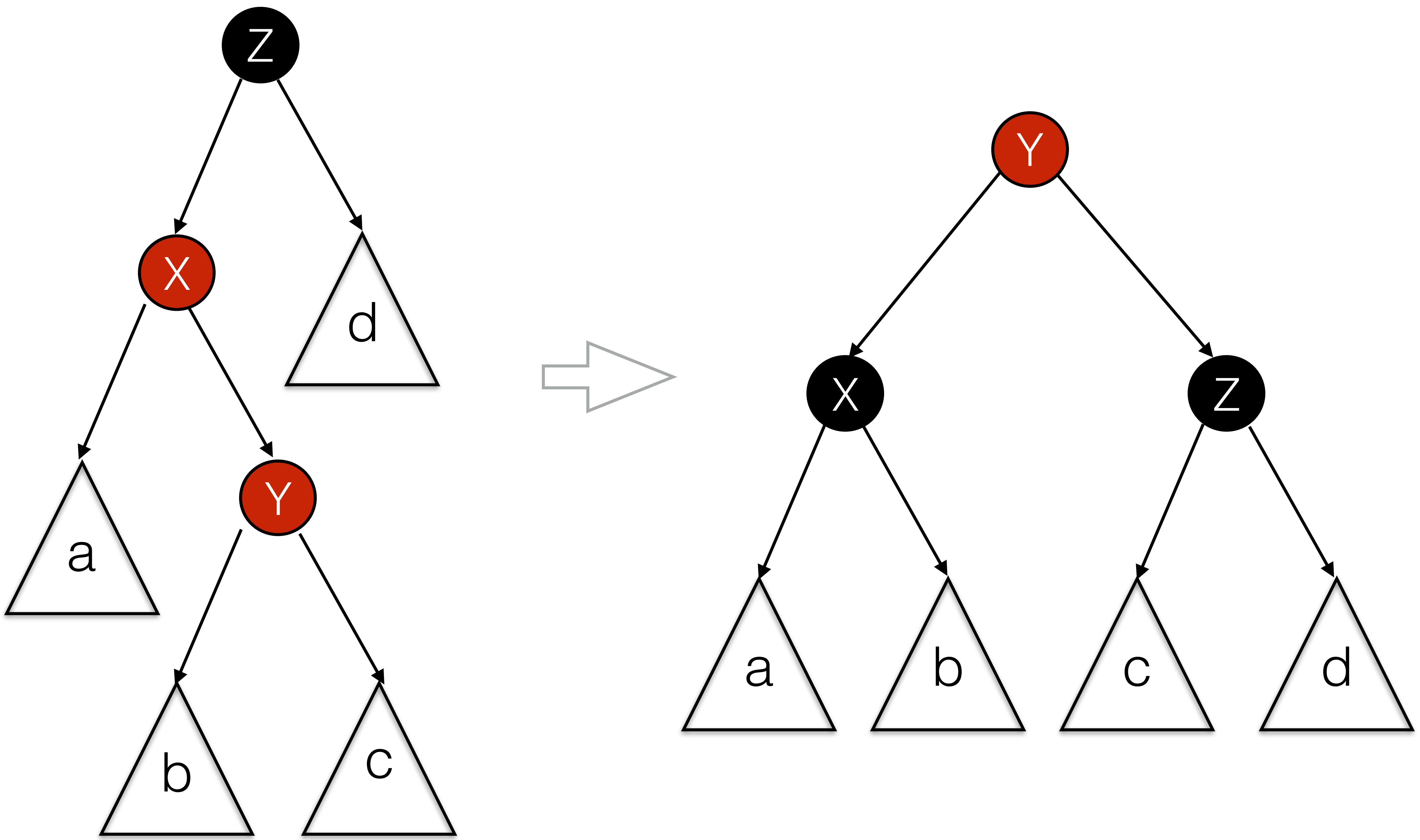
      …
    }
  }
  …
```



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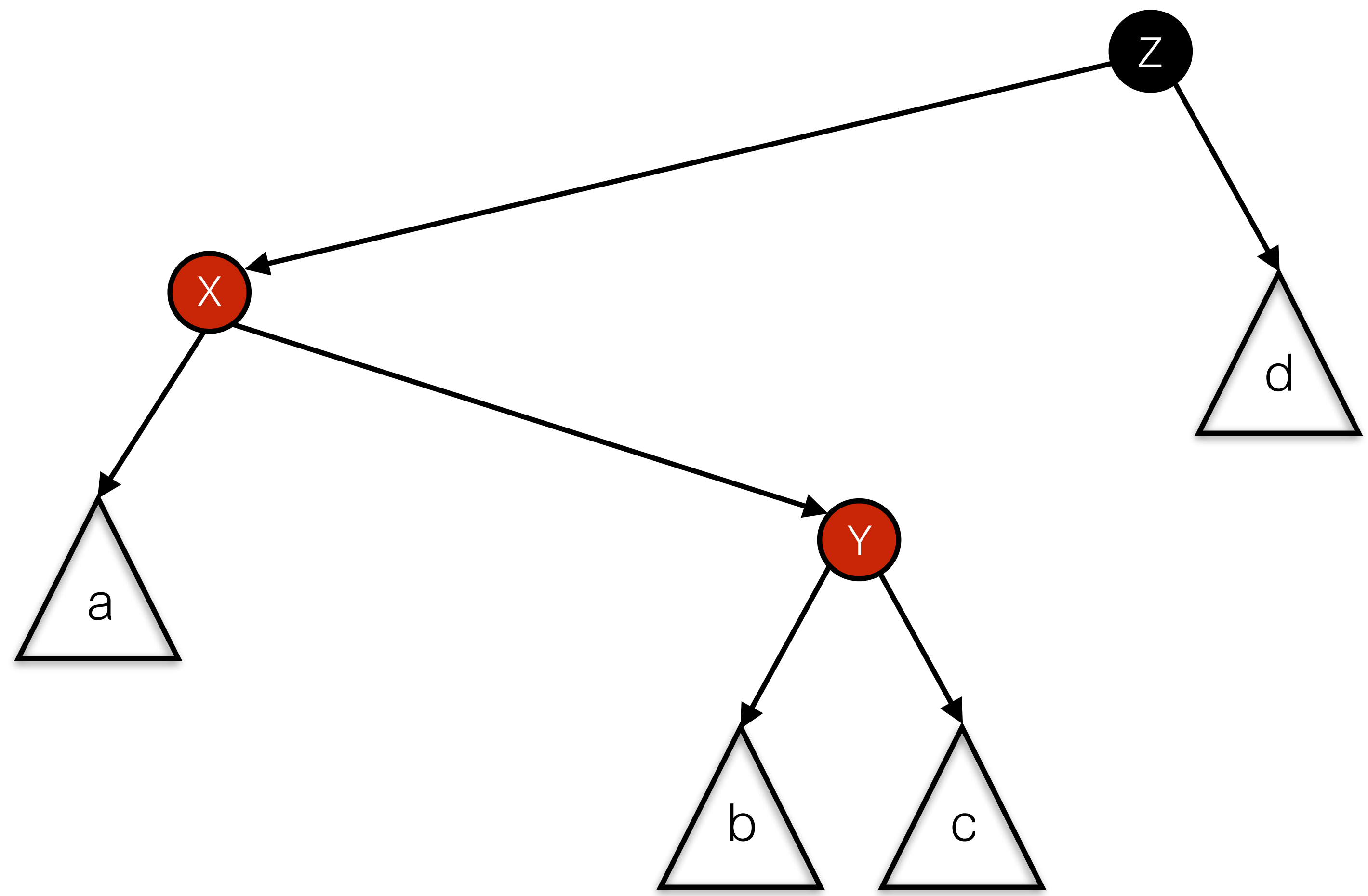


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        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>

      …
    }
}
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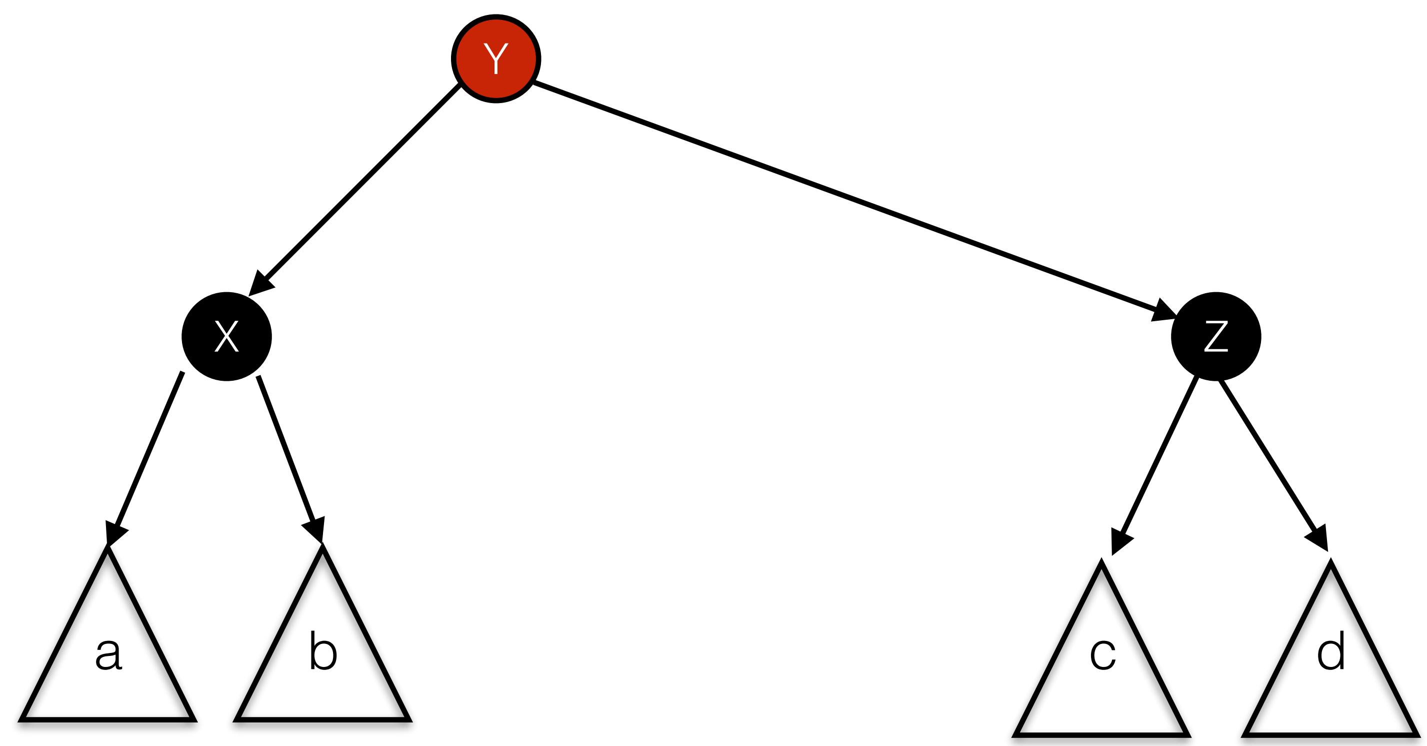


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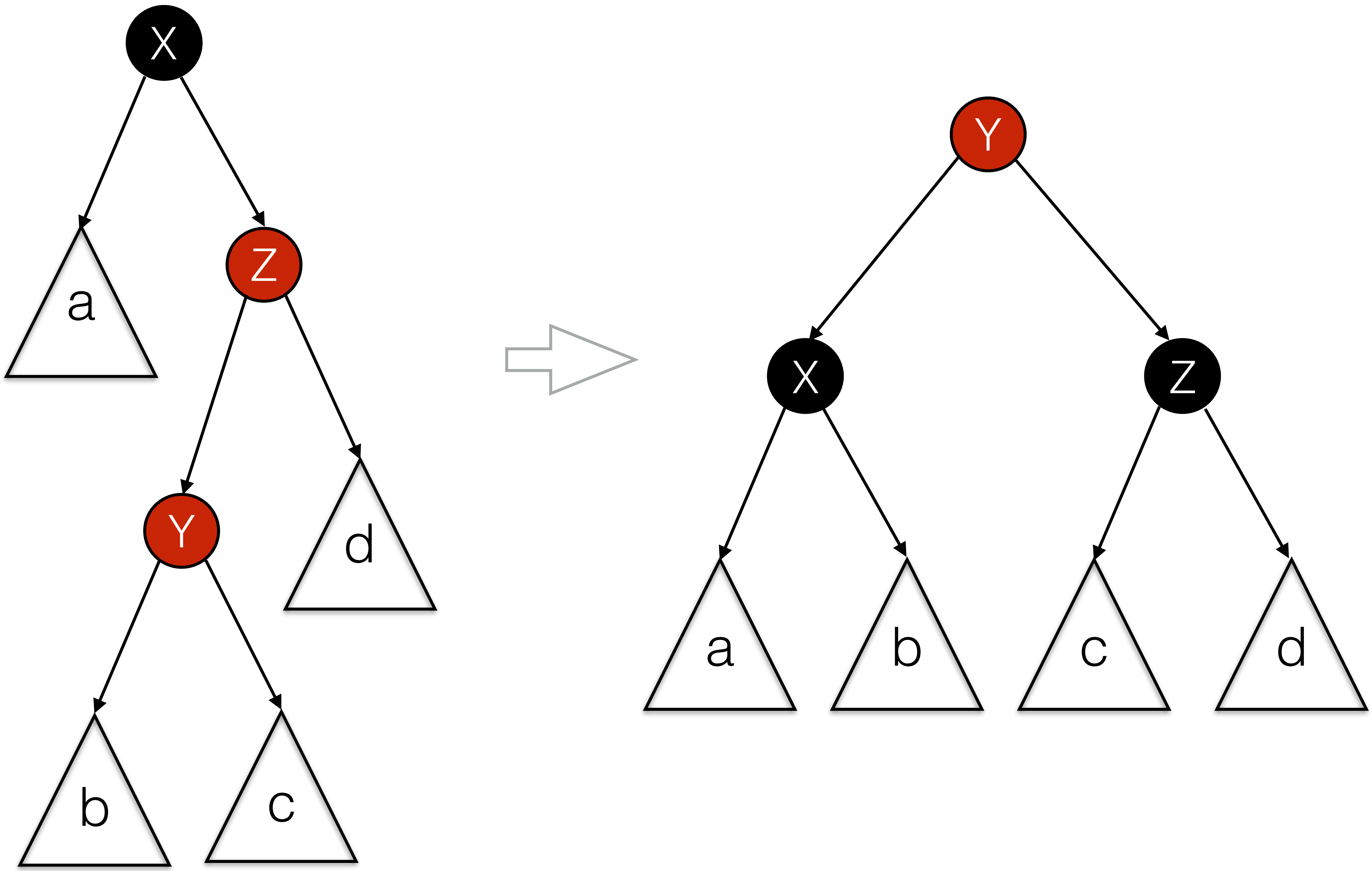
      …
    }
  }
```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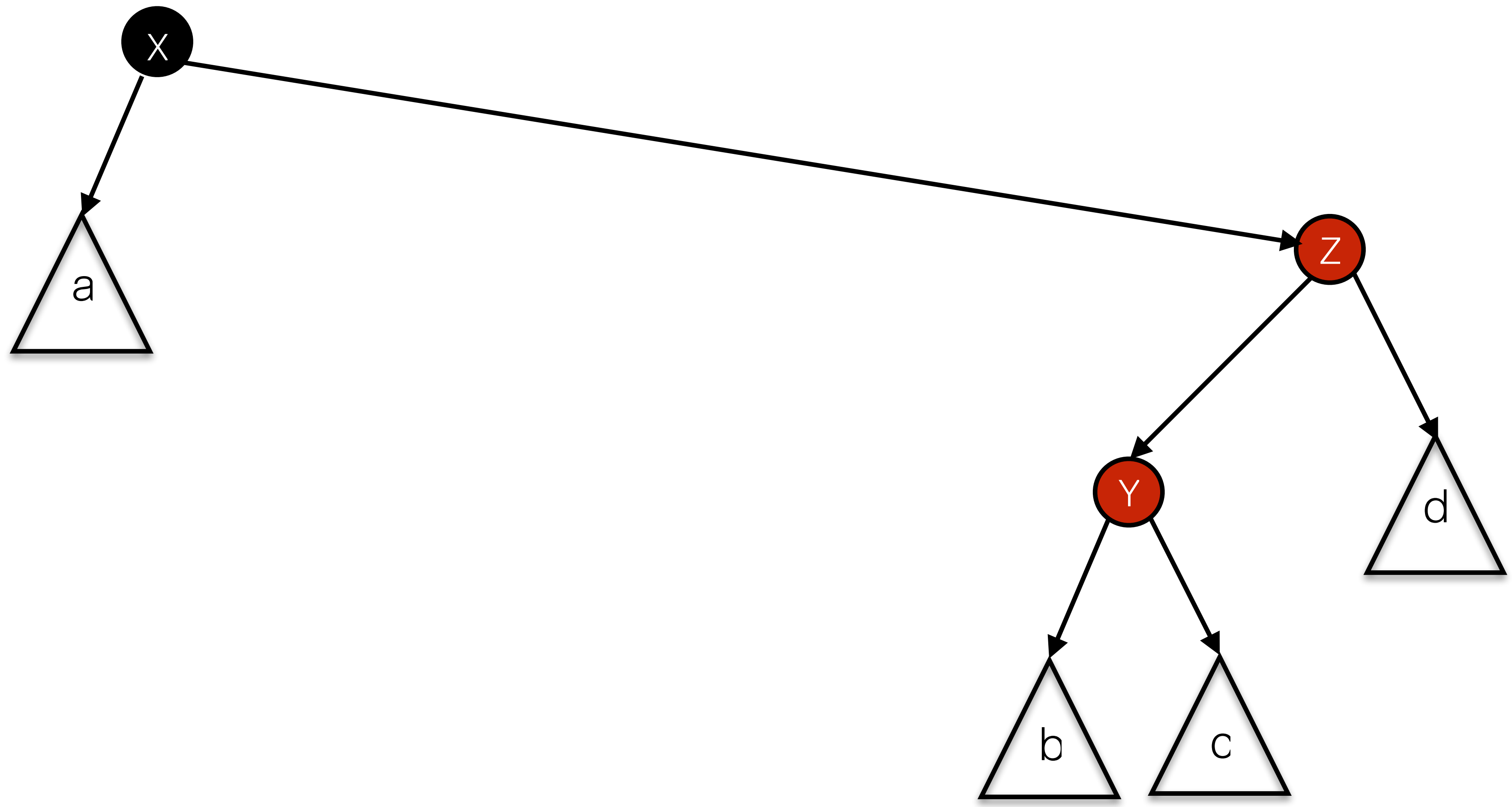


```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
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      …
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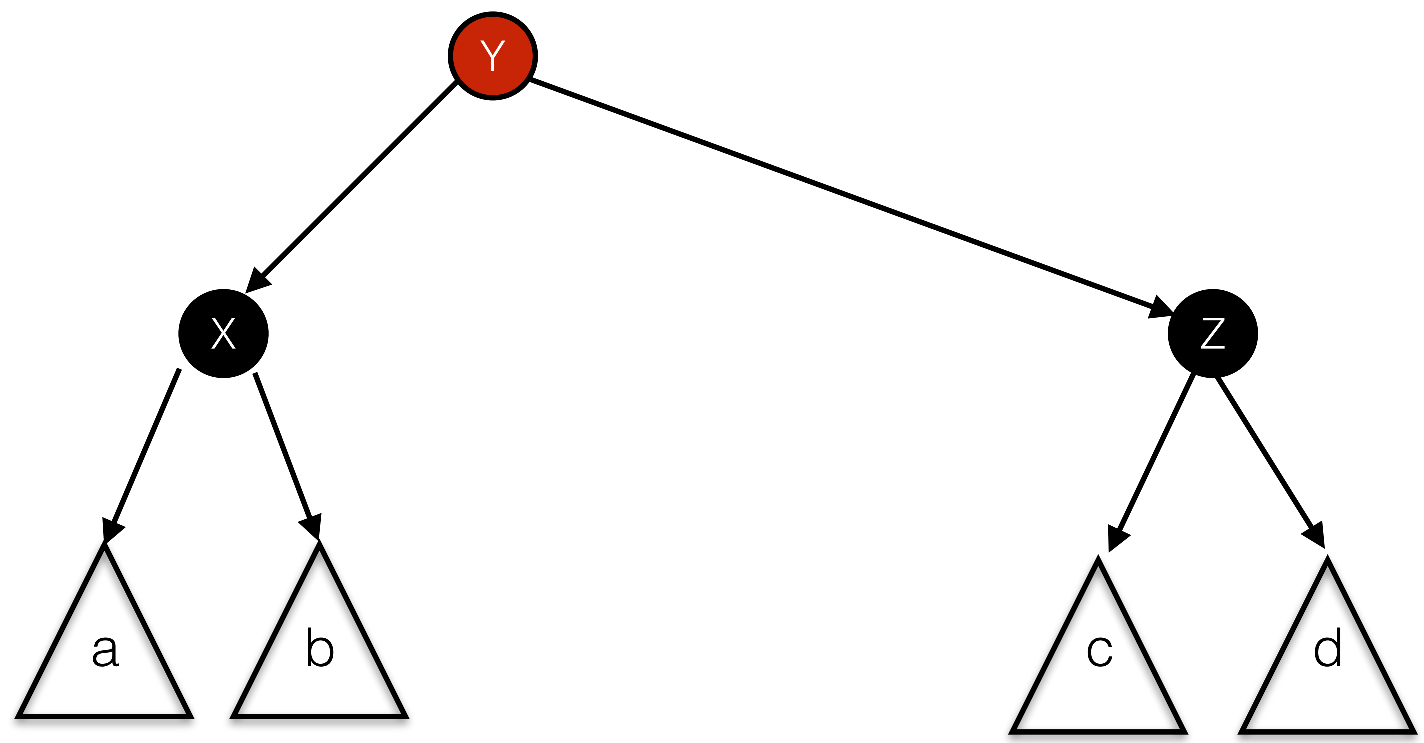


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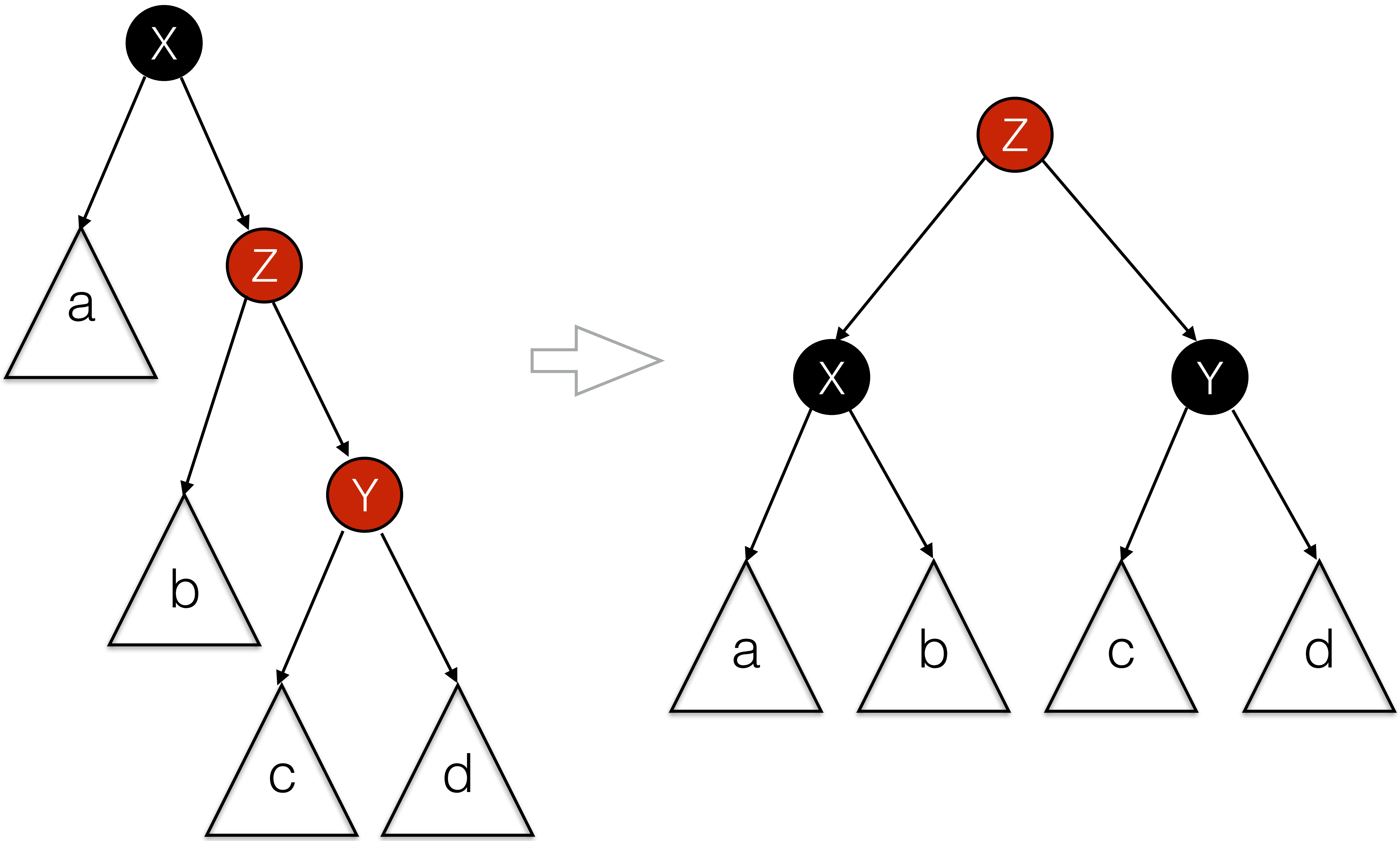

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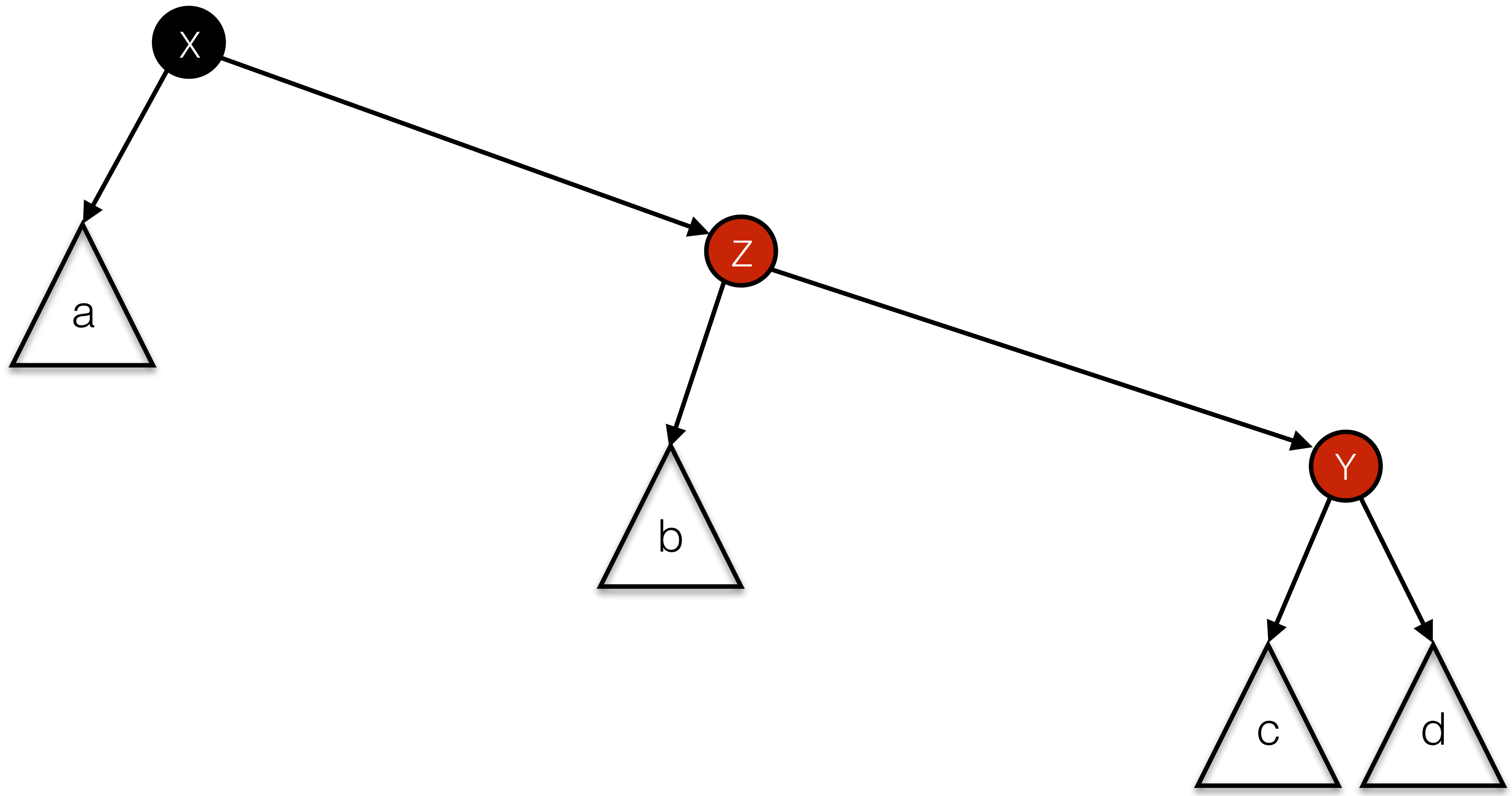


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      …
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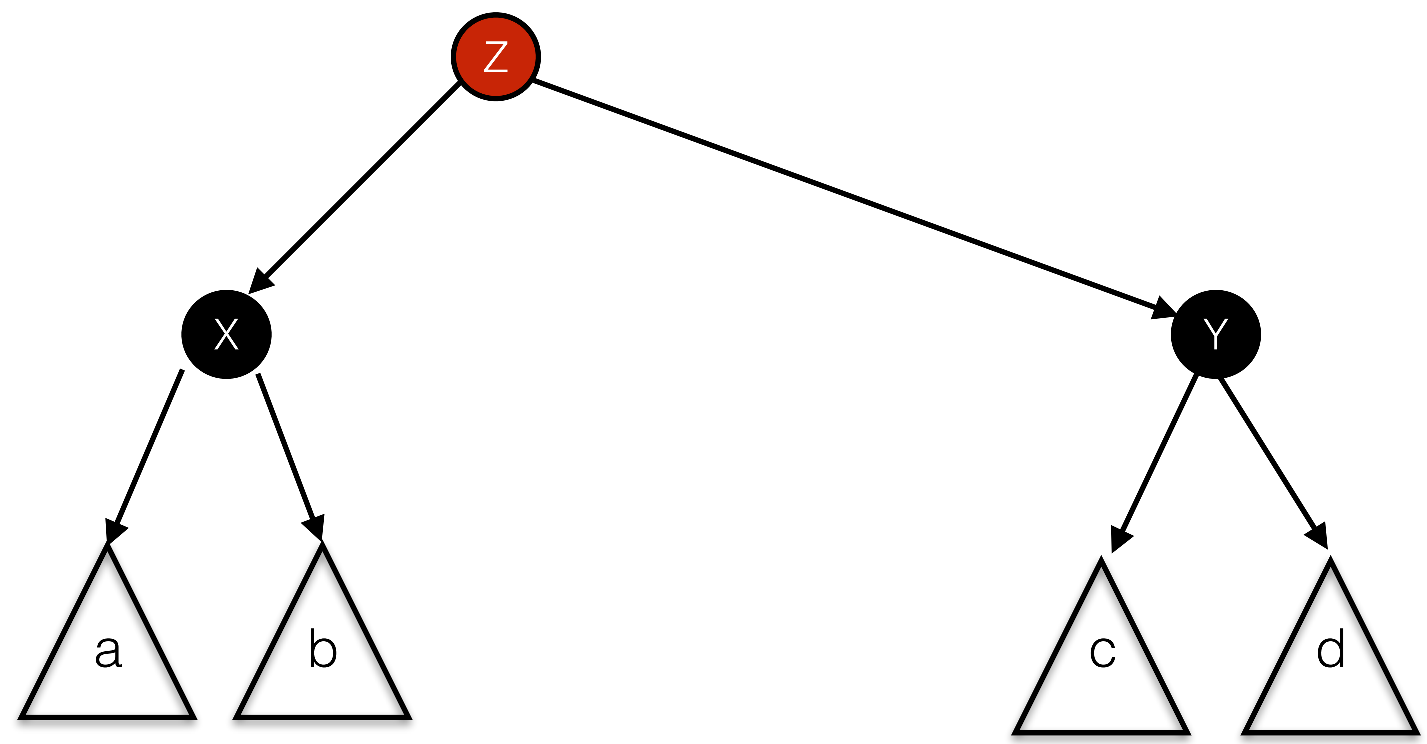
```

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      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
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      …
```

```scala
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      …
    }
  }
```

```scala
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case _ => Branch(c, l, x, r)
    }
}
```

Red-Black Trees

```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {
  …
  def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case _ => Branch(c, l, x, r)
    }
  }
  …
}
```

Unfortunately, all four consequences are syntactically identical

Red-Black Trees

```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {
  …
  def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case _ => Branch(c, l, x, r)
    }
  }
  …
}
```

In some languages (such as ML) we could factor this out with "or" patterns

Discussion

- This implementation of red-black trees is dramatically simpler than most imperative approaches:
    - Imperative approaches typically include eight cases, branching on the color of the red parent’s sibling
    - These cases help to avoid some assignment and copying in an imperative setting