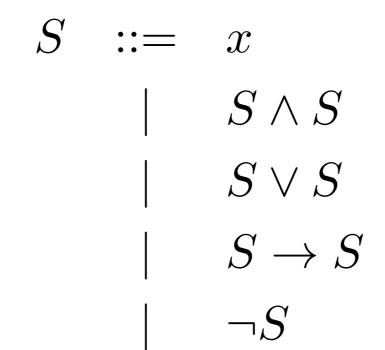
Comp 311 Functional Programming

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Mechanical Proof Checking

Syntax of Propositional Logic



Factory Methods for Construction

case object Formulas {

}

- def evar(name: String): Formula
- def and(left: Formula, right: Formula): Formula
- def or(left: Formula, right: Formula): Formula
- def implies(left: Formula, right: Formula): Formula
 def not(body: Formula): Formula

Sequents

$S* \vdash S$

Sequents

- Sequents consist of two parts:
 - The antecedents to the left of the turnstile
 - The *consequent* to the right of the turnstile
 - Example:

$$\{p, q, \neg r, p \to r\} \vdash \neg p$$

Sequents

• When the set of antecedents consists of a single formula, we often elide the enclosing braces:

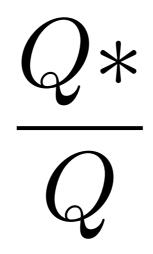
$$\{p\} \vdash p$$

• is equivalent to:

$$p \vdash p$$

 $\frac{\Gamma \vdash p \quad \Delta \vdash q}{\Gamma \cup \Delta \vdash p \land q} \text{ And-Intro}$

Inference Rules: General Form



 $\frac{\Gamma \vdash p \land q}{\Gamma \vdash p} \text{ And-Elim-Left}$

$$\frac{\Gamma \vdash p \land q}{\Gamma \vdash q} \text{ And-Elim-Right}$$

 $\frac{\Gamma \vdash p}{\Gamma \vdash p \lor q} \text{ Or-Intro-Left}$

 $\frac{\Gamma \vdash p}{\Gamma \vdash q \lor p} \text{ Or-Intro-Right}$

$\frac{\Gamma \vdash p \lor q \quad \Gamma' \cup \{p\} \vdash r \quad \Gamma'' \cup \{q\} \vdash r}{\Gamma \cup \Gamma' \cup \Gamma'' \vdash r} \text{ Or-Elim}$

$\frac{\Gamma \cup \{p\} \vdash q \quad \Gamma' \cup \{p\} \vdash \neg q}{\Gamma \cup \Gamma' \vdash \neg p}$ Neg-Intro

 $\frac{\Gamma \vdash \neg \neg p}{\Gamma \vdash p} \text{ Neg-Elim}$

 $\frac{\Gamma \cup \{p\} \vdash q}{\Gamma \vdash p \to q} \text{ Implies-Intro}$

$\frac{\Gamma \vdash p \to q \quad \Gamma' \vdash p}{\Gamma \cup \Gamma' \vdash q} \text{ IMPLIES-ELIM}$

 $\frac{}{p \vdash p} \text{ Identity}$

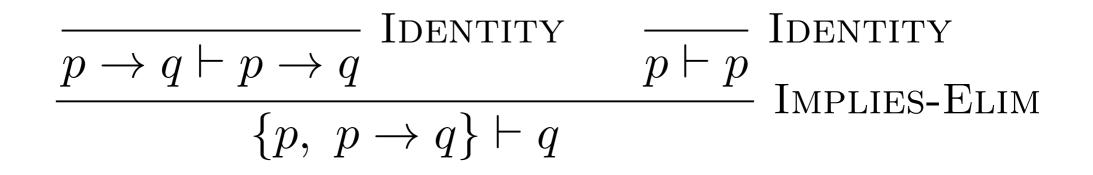
 $\overline{\Gamma \cup \{p\} \vdash p} \text{ Assumption}$

 $\frac{\Gamma \vdash p}{\Gamma \cup \{q\} \vdash p} \text{ Generalization}$

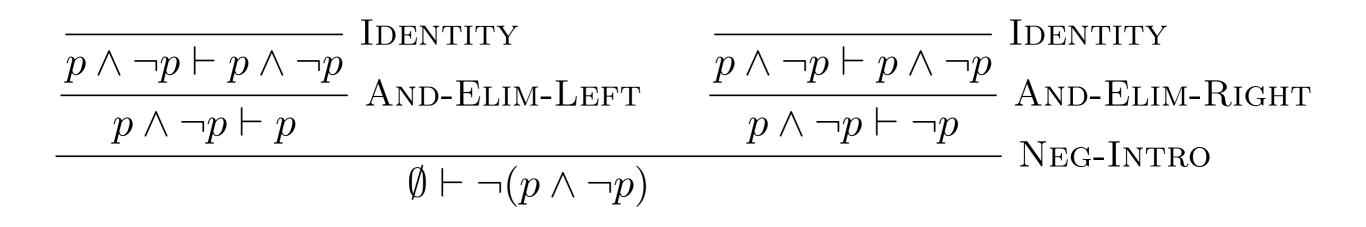
Example Proof 1

$\frac{-}{p \vdash p} \quad \text{Identity} \\ \frac{p \vdash p}{\emptyset \vdash p \to p} \quad \text{Implies-Intro}$

Example Proof 2



Example Proof 3



Rule Application

```
case object Rules {
  def identity(p: Formula): Sequent
  def assumption(s: Sequent): Sequent
  def generalization(p: Formula)(s: Sequent): Sequent
  def andIntro(left: Sequent, right: Sequent): Sequent
  def andElimLeft(s: Sequent): Sequent
  def andElimRight(s: Sequent): Sequent
  def orIntroLeft(p: Formula)(s: Sequent): Sequent
  def orIntroRight(p: Formula)(s: Sequent): Sequent
  def orElim(s0: Sequent, s1: Sequent, s2: Sequent): Sequent
  def negIntro(p: Formula)(s0: Sequent, s1: Sequent): Sequent
  def negElim(s: Sequent): Sequent
  def impliesIntro(s: Sequent): Sequent
  def impliesElim(p: Formula)(s: Sequent): Sequent
```

```
}
```

The Curry-Howard Isomorphism

Simply Typed Expressions

E ::= x
 | 0 | 1 | 2...
 | true | false
 | (x:T) => E
 | E(E)

Simple Types

T ::= Int | Boolean | T => T

E:T

0:Int

true:Boolean

 $(x:Int) \Rightarrow x : Int \Rightarrow Int$

x:Boolean

Assertions Within a Type Environment

 ${x:Boolean} \vdash x:Boolean$

Rules for Checking the Type of an Expression

 $\frac{n \in \texttt{IntLiteral}}{\Gamma \vdash \texttt{n:Int}} \xrightarrow{\text{T-Int}}$

Rules for Checking the Type of an Expression





Rules for Checking the Type of an Expression

$$\frac{\Gamma \cup \{\mathbf{x}: \mathbf{S}\} \vdash \mathbf{E}: \mathbf{T}}{\Gamma \vdash (\mathbf{x}: \mathbf{S}) => \mathbf{E} : \mathbf{S} => \mathbf{T}} \text{ T-ABS}$$

Rules for Checking the Type of an Expression

 $\frac{\Gamma \vdash E:S=>T \quad \Gamma \vdash E':S}{\Gamma \vdash E(E'):T} \quad T-APP$

Contrast with Implies-Intro For Propositional Logic

$$\frac{\Gamma \cup \{p\} \vdash q}{\Gamma \vdash p \to q} \text{ Implies-Intro}$$

$$\frac{\Gamma \cup \{\mathbf{x}: \mathbf{S}\} \vdash \mathbf{E}: \mathbf{T}}{\Gamma \vdash (\mathbf{x}: \mathbf{S}) = \mathbf{E} : \mathbf{S} = \mathbf{T}} \text{ T-ABS}$$

Contrast with Implies-Intro For Propositional Logic

$$\frac{\Gamma \cup \{p\} \vdash q}{\Gamma \vdash p \to q} \text{ Implies-Intro}$$

$$\begin{array}{c|c} \Gamma \cup \{ S \} \vdash T \\ \hline \Gamma \vdash S = T \end{array} T-ABS$$

Contrast with Implies-Elim From Propositional Logic

$$\frac{\Gamma \vdash p \to q \quad \Gamma' \vdash p}{\Gamma \cup \Gamma' \vdash q} \text{ Implies-Elim}$$

$$\frac{\Gamma \vdash E:S=T \quad \Gamma \vdash E':S}{\Gamma \vdash E(E'):T} \quad T-APP$$

Contrast with Implies-Elim From Propositional Logic

$$\frac{\Gamma \vdash p \to q \quad \Gamma' \vdash p}{\Gamma \cup \Gamma' \vdash q} \text{ Implies-Elim}$$

- We can think of the types in our simple type system as corresponding to propositions:
 - Primitive types (Boolean, Int) correspond to simple propositions (p, q)
 - Arrow types correspond to logic implication:

p -> q, (p -> (q -> r)), etc.

- For each syntactic form of expression, there is exactly one form of rule that contains that syntactic form as its result
- Example:

$$\frac{\Gamma \cup \{\mathbf{x}: \mathbf{S}\} \vdash \mathbf{E}: \mathbf{T}}{\Gamma \vdash (\mathbf{x}: \mathbf{S}) = \mathbf{E} : \mathbf{S} = \mathbf{T}} \text{ T-ABS}$$

- If we wish to use type rules to prove that an expression has a specific type
 - We can start with the expression, and apply the rules backwards:

$$\frac{\overline{\mathbf{x}: \mathbf{T} \vdash \mathbf{x}: \mathbf{T}} \quad \text{T-IDENTITY}}{\emptyset \vdash (\mathbf{x}: \mathbf{T}) \implies \mathbf{x} : \mathbf{T} \implies \mathbf{T}} \quad \text{T-ABS}$$

- While working backwards with expressions, there is only one choice at each step
- Thus a well-typed expression E entirely determines the form of the proof that E:T
- But the proof of E:T in our type system is equivalent to a proof of T in propositional logic

- So, E effectively encodes a proof of type T, thought of as a proposition
- Checking the type T of an expression E is equivalent to proving the validity of T

The Curry-Howard Isomorphism

- This deep correspondence between types and logical assertions is known as the *Curry-Howard Isomorphism*
- This correspondence goes far beyond just propositional logic, extending to predicate calculus, modal logic, etc.
- This leads to the surprising result that the arrow in arrow types is really just the implication symbol from propositional logic!