

# Comp 311

# Functional Programming

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# Partially Applied Functions

- If we want to pass a function as an argument, but supply some of the arguments to the function ourselves, we can wrap an application to the function in a function literal:

```
map(x => x + 1, xs)
```

# Partially Applied Functions

- If we want to pass a function as an argument, but supply some of the arguments to the function ourselves, we can wrap an application to the function in a function literal:

```
map(x => x + 1, xs)
```

which is equivalent to

```
map(_ + 1, xs)
```

# Partially Applied Functions

- **Eta Expansion:** Wrapping a function in function literal that takes all of the arguments of `f` and immediately calls `f` with those arguments

`(x:Int) => square(x)`

is equivalent to

`square`

# Mapping a Computation Over a List

We can use eta expansion to pass operators  
as arguments:

```
map(x => -x, xs)
```

# Mapping a Computation Over a List

We can use eta expansion to pass operators  
as arguments:

```
map(-_, xs)
```

# Returning Functions as Values

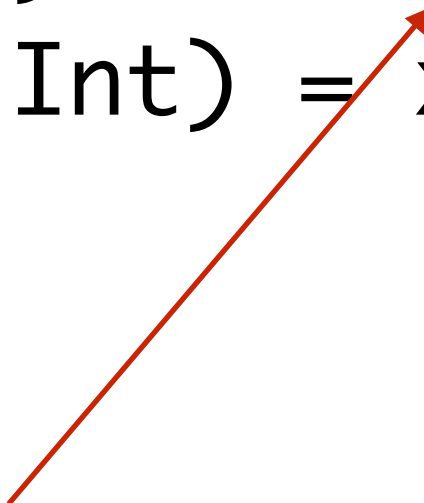
# We Can Define Functions That Return Other Functions as Values

```
def add(x: Int): Int => Int = {  
  def addX(y: Int) = x + y  
  addX  
}
```



# We Can Define Functions That Return Other Functions as Values

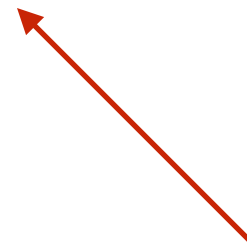
```
def add(x: Int): Int => Int = {  
  def addX(y: Int) = x + y  
  addX  
}
```



The explicit return type is needed because Scala type inference assumes an unapplied function is an error

# We Can Define Functions That Return Other Functions as Values

```
def add(x: Int) = {  
  def addX(y: Int) = x + y  
  addX _  
}
```



Alternatively, we can eta-expand addX to assure the type checker that we really do intend to return a function

# We Can Define Functions That Return Other Functions as Values

```
def add(x: Int) = {  
  def addX(y: Int) = x + y  
  addX _  
}
```

An underscore outside of parentheses in a function application denotes the entire tuple of arguments passed to the function

# We Can Define Functions That Return Other Functions as Values

```
def add(x: Int) = x + (_: Int)
```

We can instead define `add` by *partially* eta-expanding the `+` operator. But then we need to annotate the second operand with a type.

# Aside: Type Annotations

- In general, an expression annotated with a type is itself an expression:

`expr: Type`

- If the static type of `expr` is a subtype of `Type`, then the type of `expr:Type` is `Type`

# Partial Eta-Expansion

- We can partially eta-expand any function, but we need to annotate the argument types:

```
def reduce0 =  
  reduce(0, _: (Int, Int) => Int, _: List)
```

# Derivatives

```
def derivative(f: Double => Double, dx: Double) =  
  (x: Double) =>  
    (f(x + dx) - f(x)) /  
    dx
```

# Derivatives

```
def f(x: Double) = x * x  
def Df = derivative(f, 0.00001)
```

$f(4) \mapsto 16$

$Df(4) \mapsto 8.00000999952033$

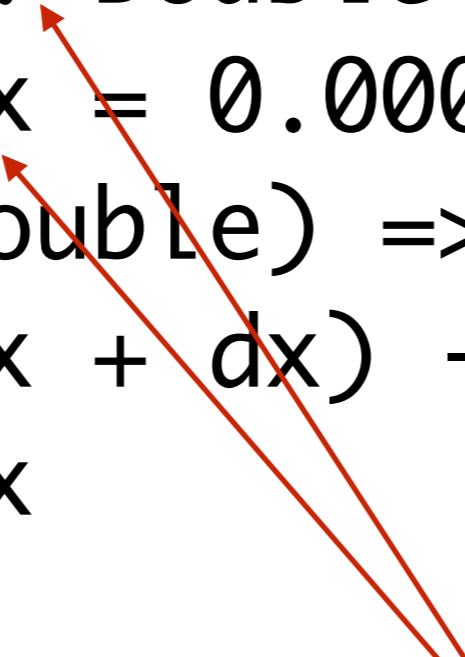


# Encapsulating dx

```
def D(f: Double => Double) = {  
  val dx = 0.00001  
  (x: Double) =>  
    (f(x + dx) - f(x)) /  
      dx  
}
```

# Encapsulating dx

```
def D(f: Double => Double) = {  
  val dx = 0.00001  
  (x: Double) =>  
    (f(x + dx) - f(x)) /  
    dx  
}
```



Our returned function “remembers”  
these values

# Applying a Derivative

```
def D(f: Double => Double) = {  
  val dx = 0.00001  
  (x: Double) =>  
    (f(x + dx) - f(x)) /  
    dx  
}
```

$D(f)(4) \mapsto$

$D((x: Double) => x * x)(4) \mapsto$

# Applying a Derivative

$D((x: \text{Double}) \Rightarrow x * x)(4) \mapsto$

`{val dx = 0.00001`

`(x: Double) =>`

`((x: Double) => x * x)(x + dx) -`

`(x: Double) => x * x)(x)) /`

`dx }`(4)  $\mapsto$

# Applying a Derivative

```
{(x: Double) =>
  ((x: Double) => x * x)(x + 0.00001) -
  ((x: Double) => x * x)(x)) /
  0.00001}(4) ↪
```

```
((x: Double) => x * x)(4 + 0.00001) -
  ((x: Double) => x * x)(4)) /
  0.00001 ↪
```

We must be careful to substitute only corresponding occurrences of x

# Applying a Derivative

$$\frac{((x: \text{Double}) \Rightarrow x * x)(4 + 0.00001) - ((x: \text{Double}) \Rightarrow x * x)(4)}{0.00001} \mapsto$$

$$\frac{((x: \text{Double}) \Rightarrow x * x)(4.00001) - ((x: \text{Double}) \Rightarrow x * x)(4)}{0.00001} \mapsto$$

$$\frac{(4.00001 * 4.00001) - (4 * 4)}{0.00001} \mapsto$$

# Applying a Derivative

$$\frac{((4.00001 * 4.00001) - (4 * 4))}{0.00001} \mapsto$$

$$\frac{(16.000080000099995 - 16)}{0.00001} \mapsto$$

$$8.00000999952033E-5 / 0.00001 \mapsto$$

$$8.00000999952033$$

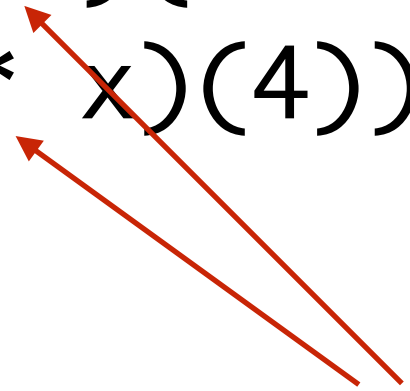
# Safe Substitution



# Applying a Derivative

```
{(x: Double) =>
  ((x: Double) => x * x)(x + 0.00001) -
  ((x: Double) => x * x)(x)) /
  0.00001}(4) ↦
```

```
((x: Double) => x * x)(4 + 0.00001) -
  ((x: Double) => x * x)(4)) /
  0.00001
```



In cases like this one, we can avoid accidental variable capture by selective renaming

# Safe Substitution

(a.k.a. Alpha Renaming)

- We can ensure we never accidentally substitute the wrong parameters by automatically renaming constants, functions, and parameters with *fresh* names
  - A fresh name must not capture a name referred to in the scope of a parameter
  - A fresh name must not be captured by a name in an enclosing scope

# Applying a Derivative

```
{(x: Double) =>  
  ((y: Double) => y * y)(x + 0.00001) -  
  ((z: Double) => z * z)(x)) /  
  0.00001}(4) ↦
```

```
((y: Double) => y * y)(4 + 0.00001) -  
  ((z: Double) => z * z)(4)) /  
  0.00001
```

# Function Equivalence

- Now we have seen the three forms of function equivalence stipulated by the Lambda Calculus:
  - Alpha Renaming: Changing the names of a function's parameters does not affect the meaning of the function
  - Beta Reduction: To apply a function to an argument, reduce to the body of the function, substituting occurrences of the parameter with the corresponding argument
  - Eta Equivalence: Two functions are equivalent iff they are *extensionally equivalent*: They give the same results for all arguments

# Parametric Types

# Parametric Types

- We have defined two forms of lists: lists of ints and lists of shapes
- Many computations useful for one are useful for the other:
  - Map, reduce, filter, etc.
- It would be better to define lists and their operations once for all of these cases

# Parametric Types

- Higher-order functions take functions as arguments and return functions as results
- Likewise, *parametric types*, a.k.a., *a generic types*, takes types as arguments and return types as results

# Parametric Lists

- Every application of this parametric type to an argument yields a new type:

```
abstract class List[T] {  
  def ++(ys: List[T]): List[T]  
}
```



# Parametric Lists

- Every application of this parametric type to an argument yields a new type:

```
abstract class List[T <: Any] {  
  def ++(ys: List[T]): List[T]  
}
```



- We augment the declarations of type parameters to permit an upper bound on all instantiations of a parameter
- By default, the bound is **Any**

# Syntax of Parametric Class Definitions

```
<modifiers> class C[T1 <: N, ..., TN <: N] extends N {  
  <ordinary class body>  
}
```

- We denote “naked” type parameters as **T1**, **T2**, etc.
- We denote all other types with **N**, **M**, etc.

# Syntax of Parametric Class Definitions

```
<modifiers> class C[T1 <: N, ..., TN <: N] extends N {  
  <ordinary class body>  
}
```

- Declared type parameters T1, ..., TN are in scope throughout the entire class definition, including:
  - The bounds of type parameters
  - The **extends** clause
- Object definitions must not be parametric

# Parametric Lists

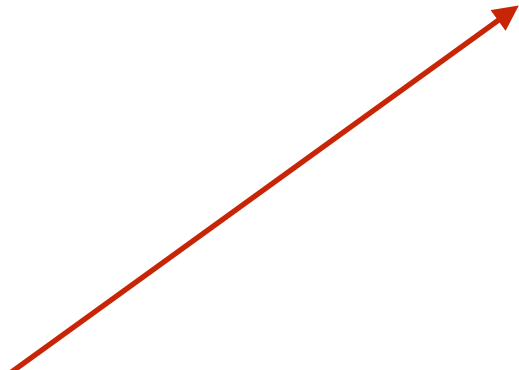
- Every application of this parametric type yields a new type:

```
List[Int]  
List[String]  
List[List[Double]]  
etc.
```

# Parametric Lists

- Every application (a.k.a., *instantiation*) of this parametric type yields a new type:

```
abstract class List[T] {  
    def ++(ys: List[T]): List[T]  
}
```



Note that our parametric type can be instantiated with type parameters, including its own!


# Parametric Lists

```
case class Empty[S]() extends List[S] {  
  def ++(ys: List[S]) = ys  
}
```

```
case class Cons[T](head: T, tail: List[T]) extends List[T] {  
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)  
}
```

# Parametric Lists

```
case class Empty[S]() extends List[S] {  
  def ++(ys: List[S]) = ys  
}  
  
case class Cons[T](head: T, tail: List[T]) extends List[T] {  
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)  
}
```



Our definition requires a separate type `Empty[S]` for every instantiation of `S`. Thus we must define `Empty` as a class rather than an object.

# Type Environments

- To explain how to type check expressions in the context of parametric types, we must introduce the notion of *environments*
- We define a type parameter environment to hold a collection of zero or more type parameter declarations with their bounds
- Type environments can be extended with more declarations



# Type Checking a Class Definition

- To type check a parametric class definition:
  - Check the declarations of the class in a new type parameter environment that extends the enclosing environment with all its type parameters

# Type Checking a Function Definition

- To type check a function definition in environment  $E$ :
  - Check that the types of all parameters are *well-formed*
  - Find the type of the body of the function, substituting occurrences of parameters with their types
  - Ensure that the type of the body is a subtype of the declared return type (in environment  $E$ )

# Well-Formedness of Types

- A type is well-formed in environment  $E$  iff:
  - If it is a well-defined non-parametric type
  - It is a type parameter  $T$  in environment  $E$
  - It is an instantiation of a defined parametric type and:
    - All of its type arguments are well-formed types in  $E$
    - All of its type arguments respect the bounds on their corresponding type parameters

# Subtyping With Environments

- It is non-sensical to compare types in separate type environments:

```
case class Empty[S]() extends List[S] {  
  def ++(ys: List[S]) = ys  
}
```

```
case class Cons[T](head: T, tail: List[T]) extends List[T] {  
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)  
}
```

- Is S a subtype of T?

# Subtyping With Environments

- We must modify our subtyping rules to refer to an environment  $E$ :
  - $S <: S$  in  $E$
  - If  $S <: T$  in  $E$  and  $T <: U$  in  $E$  then  $S <: U$  in  $E$

# Subtyping With Environments

- If:
  - `class C[T1, ..., TN] extends D[U1, ...UM]`
  - and  $X_1, \dots, X_N$  are well-formed in  $E$
  - then `C[X1, ...XN] <: D[U1, ..., UM] [T1 ↦ X1, ..., TN ↦ XN]` in  $E$

# Subtyping With Environments

- If:
  - `class C[T1, ..., TN] extends D[U1, ...UM]`
  - and  $X_1, \dots, X_N$  are well-formed in  $E$
  - then  $C[X_1, \dots, X_N] <: D[U_1, \dots, U_M][T_1 \mapsto X_1, \dots, T_N \mapsto X_N]$  in  $E$

We use this notation to indicate safe substitution of  $T_1$  for  $X_1$ ,  
...  $T_N$  for  $X_N$  in  $D[U_1, \dots, U_M]$

# Covariance

- Can one instantiation of a parametric type be a subtype of another?
- Currently our rules allow this only in the reflexive case:

`List[Int] <: List[Int] in E`



# Covariance

- It would be useful to allow some instantiations to be subtypes of another
- For example, we would like it to be the case that:

```
List[Int] <: List[Any]
```

# Covariance

- In general, we say that a parametric type  $C$  is covariant with respect to its type parameter  $S$  if:

$$S <: T \text{ in } E$$

implies

$$C[S] <: C[T] \text{ in } E$$

- We must be careful that such relationships do not break the soundness of our type system

# Covariance

- For a parametric type such as:

```
abstract class List[T <: Any] {  
    def ++(ys: List[T]): List[T]  
}
```

- And types  $S$  and  $T$ , such that  $S <: T$  in some environment  $E$ :
- What must we check about the body of class `List` to allow for `List[S] <: List[T]` in  $E$ ?

# Covariance

- Consider instantiations for types `String` and `Any`:

```
abstract class List[Any] {  
  def ++(ys: List[Any]): List[Any]  
}
```

```
abstract class List[String] {  
  def ++(ys: List[String]): List[String]  
}
```

# Covariance

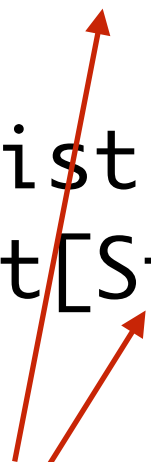
- If these were ordinary classes connected by an **extends** class:
- We would need to ensure that the overriding definition of **++** in class **List[String]** was compatible with the overridden definition in **List[Any]**

# Covariance

```
abstract class List[Any] {  
  def ++(ys: List[Any]): List[Any]  
}  
abstract class List[String] extends List[Any] {  
  def ++(ys: List[String]): List[String]  
}
```

# Covariance

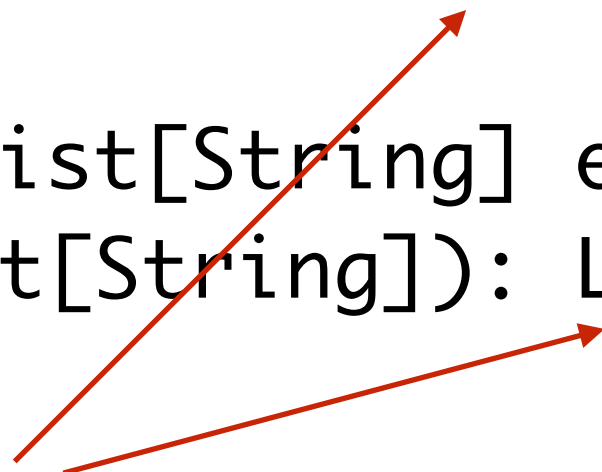
```
abstract class List[Any] {  
  def ++(ys: List[Any]): List[Any]  
}  
abstract class List[String] extends List[Any] {  
  def ++(ys: List[String]): List[String]  
}
```



But if `List[String] <: List[Any]` in E  
then this is not a valid override

# Covariance

```
abstract class List[Any] {  
  def ++(ys: List[Any]): List[Any]  
}  
abstract class List[String] extends List[Any] {  
  def ++(ys: List[String]): List[String]  
}
```



On the other hand, the return types  
are not problematic



# Covariance

- From our example, we can glean the following rule:
  - We allow a parametric class  $C$  to be covariant with respect to a type parameter  $T$  so long as  $T$  does not appear in the types of the method parameters of  $C$

# Covariance

```
abstract class List[+T] {}
```

- We stipulate that a parametric type is covariant in a parameter **T** by prefixing a **+** at the definition of **T**
- (We will return to our definition of **append** later)

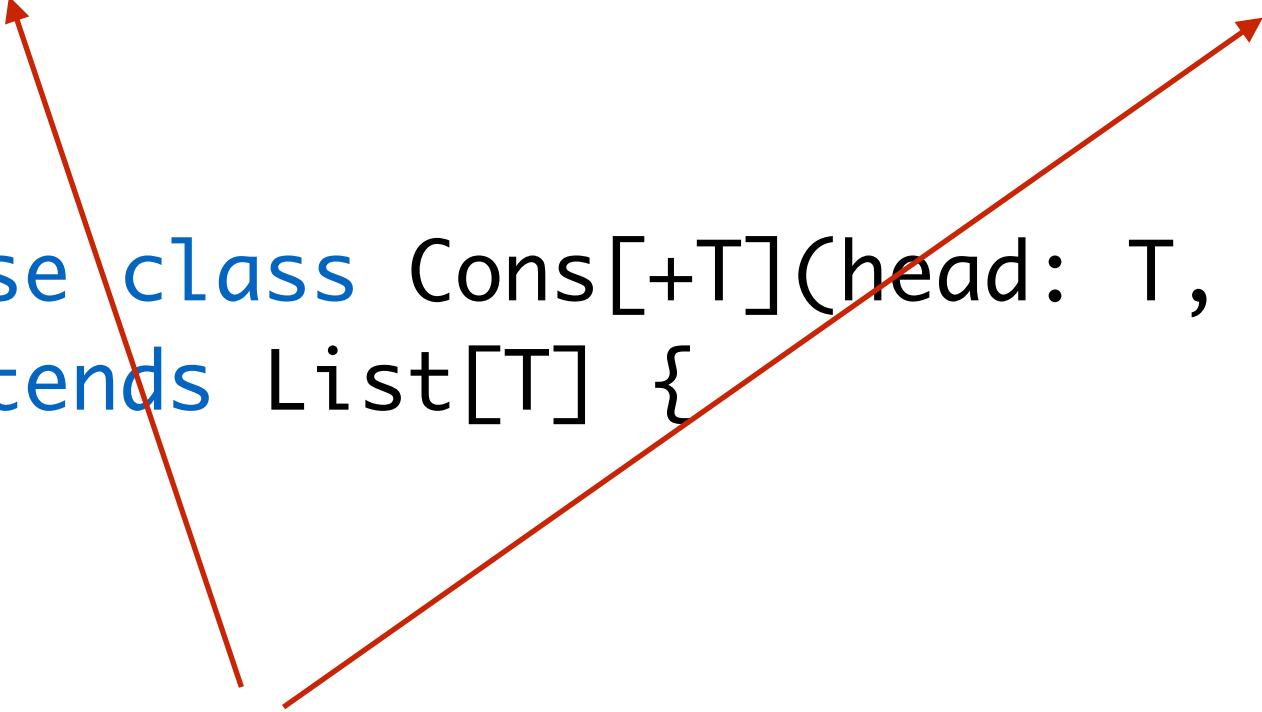
# Covariance

```
case object Empty extends List[Nothing] {  
}
```

```
case class Cons[+T](head: T, tail: List[T])  
extends List[T] {  
}
```

# Covariance

```
case object Empty extends List[Nothing] {  
}  
  
case class Cons[+T](head: T, tail: List[T])  
extends List[T] {  
}
```



*Now we can define Empty as an object that extends the bottom of the List types*

# Covariance and Append

- The problem with our original declaration of `append` was that it was not general enough:
  - There is no reason to require that we always append lists of identical type
  - Really, we can append a `List[S]` for any supertype of our `List[T]`
  - The result will be of type `List[S]`

# Lower Bounds on Type Parameters

- Thus far, we have allowed type parameters to include upper bounds:

$$T <: S$$

- They can also include lower bounds:

$$T >: U$$

- Or they can include both:

$$T >: S <: U$$

# Parametric Functions

- Just as we can add type parameters to a class definition, we can also add them to a function definition
- The type parameters are in scope in the header and body of the function

# Covariance and Append

```
abstract class List[+T] {  
  def ++[S >: T](ys: List[S]): List[S]  
}  
  
case object Empty extends List[Nothing] {  
  def ++[S](ys: List[S]) = ys  
}  
  
case class Cons[+T](head: T, tail: List[T])  
extends List[T] {  
  def ++[S >: T](ys: List[S]) = Cons(head, tail ++ ys)  
}
```



# Map Revisited

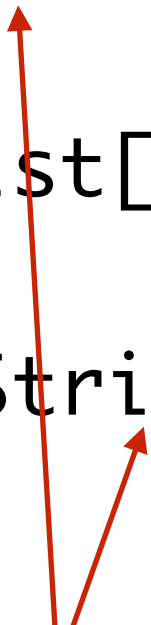
```
abstract class List[+T] {  
  ...  
  def map[U](f: T => U): List[U]  
}
```

*Why is this occurrence of T acceptable?*



# We Consider Specific Instantiations

```
abstract class List[Any] {  
  ...  
  def map[U](f: Any => U): List[U]  
}  
abstract class List[String] {  
  ...  
  def map[U](f: String => U): List[U]  
}
```



*Then List[String] is an acceptable subtype of List[Any]  
provided that  $(String \Rightarrow U) >: (Any \Rightarrow U)$   
which requires that  $String <: Any$ .*

# Generalizing Our Rules

- In our example, type parameter `T` occurs as the parameter of an arrow type:
  - $(\text{String} \Rightarrow U) >: (\text{Any} \Rightarrow U)$  in `E` provided:
    - $\text{String} <: \text{Any}$  in `E`
    - $U <: U$  in `E`
  - So subtype  $\text{List}[\text{String}] <: \text{List}[\text{Any}]$  is permitted

# To Check Variance, We Annotate Each Type Position With A *Polarity*

- Recursively descend a class definition:
  - At top level, all positions are positive
  - Polarity is flipped at method parameter positions
  - Polarity is flipped at method type parameter positions
  - Polarity is flipped at arrow type parameter positions

# Annotating Polarity

```
abstract class List[+T] {  
  def ++[S- >: T+](ys: List[S-]): List[S+]  
  def map[U-](f: T+ => U-): List[U+]  
}
```

# We Generalize Our Rules for Checking Variance As Follows

- Covariant type parameters (declared with `+`) are allowed to occur only in positive locations
- Type parameters with no annotation are allowed to be used in all locations

# Contravariance

# Contravariance

- In general, we say that a parametric type  $C$  is contravariant with respect to its type parameter  $S$  if:

$$S <: T \text{ in } E$$

implies

$$C[T] <: C[S] \text{ in } E$$

- We must be careful that such relationships do not break the soundness of our type system



# Contravariance

- Syntactically, contravariant type parameter declarations are annotated with a minus sign:

```
case class F[-A, +B]
```

# To Check Variance, We Annotate Each Type Location With A *Polarity*

- Recursively descend a class definition:
  - At top level, all locations are positive
  - Polarity is flipped at method parameter positions
  - Polarity is flipped at method type parameter positions
  - Polarity is flipped at arrow type parameter positions
  - Polarity is flipped at positions of contravariant type parameters

# Annotating Polarity

```
abstract class List[+T] {  
  def ++[S- >: T+](ys: List[S-]): List[S+]  
  def map[U-](f: T+ => U-): List[U+]  
}
```

# We Generalize Our Rules for Checking Variance As Follows

- Covariant type parameters (declared with +) are allowed to occur only in positive locations
- Type parameters with no annotation are allowed to be used in all locations
- Contravariant type parameters are allowed to occur only in negative locations

# An Example of How We Might Use Contravariant Type Parameters

```
abstract class Function1[-S,+T] {  
    def apply(x:S): T  
}
```

# Map Revisited

```
case object Empty extends List[Nothing] {  
  ...  
  def map[U](f: Nothing => U) = Empty  
}
```

# Map Revisited

```
case class Cons[+T](head: T, tail: List[T])
extends List[T] {
  ...
  def map[U](f: T => U) =
    Cons(f(head), tail.map(f))
}
```