Comp 311
Functional Programming

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Streams

Streams

- a form of “lazy” sequence
- inspired by signal-processing (e.g. digital circuits)
  - Components accept *streams* of signals as input, transform their input, and produce streams of signals as outputs

Streams

```scala
abstract class Stream[+T] {
  def head(): T
  def tail(): Stream[T]
  def map[S](f: T => S): Stream[S]
  def flatMap[S](f: T => Stream[S]): Stream[S]
  def ++[S >: T](that: Stream[S]): Stream[S]
  def withFilter(f: T => Boolean): Stream[T]
  def nth(n: Int): T
}
```

Streams

```
case object NilStream extends Stream[Nothing] {
  def head() = throw new Error()
  def tail() = throw new Error()
  def map[S](f: Nothing => S): Stream[S] = NilStream
  def flatMap[S](f: Nothing => Stream[S]): Stream[S] =
    NilStream
  def ++[S >: Nothing](that: Stream[S]) = that
  def withFilter(f: Nothing => Boolean) = NilStream
  def nth(n: Int) = throw new Error()
}
```

Streams

```
case class ConsStream[+T](head: T, lazyTail: () => Stream[T])
extends Stream[T] {
  def tail = lazyTail()
  def map[S](f: T => S): Stream[S] =
    ConsStream(f(head), () => (tail map f))
  def flatMap[S](f: T => Stream[S]): Stream[S] =
    f(head) ++ tail.flatMap(f)
  def ++[S >: T](that: Stream[S]): Stream[S] =
    ConsStream(head, () => tail ++ that)
  …
}
```

Streams

```
case class ConsStream[+T](head: T, lazyTail: () => Stream[T])
extends Stream[T] {
  …
  def withFilter(f: T => Boolean) = {
    if (f(head)) ConsStream(head, () => tail.withFilter(f))
    else tail.withFilter(f)
  }
  def nth(n: Int) = {
    require (n >= 0)
    if (n == 0) head
    else tail.nth(n - 1)
  }
}
```

Streams

```
def range(low: Int, high: Int): Stream[Int] =
  if (low > high) NilStream
  else ConsStream(low, () => range(low + 1, high))
```

Streams

```
def intsFrom(n: Int): Stream[Int] =
  ConsStream(n, () => intsFrom(n + 1))
```

Streams

```
val nats = intsFrom(0)
```

Streams

```
def fibGen(a: Int, b: Int): Stream[Int] =
  ConsStream(a, () => fibGen(b, a + b))
```

Streams

```
val fibs = fibGen(0, 1)
```

Streams

```
def push(x: Int, ys: Stream[Int]) = {
  ConsStream(x, () => ys)
}
```

Streams

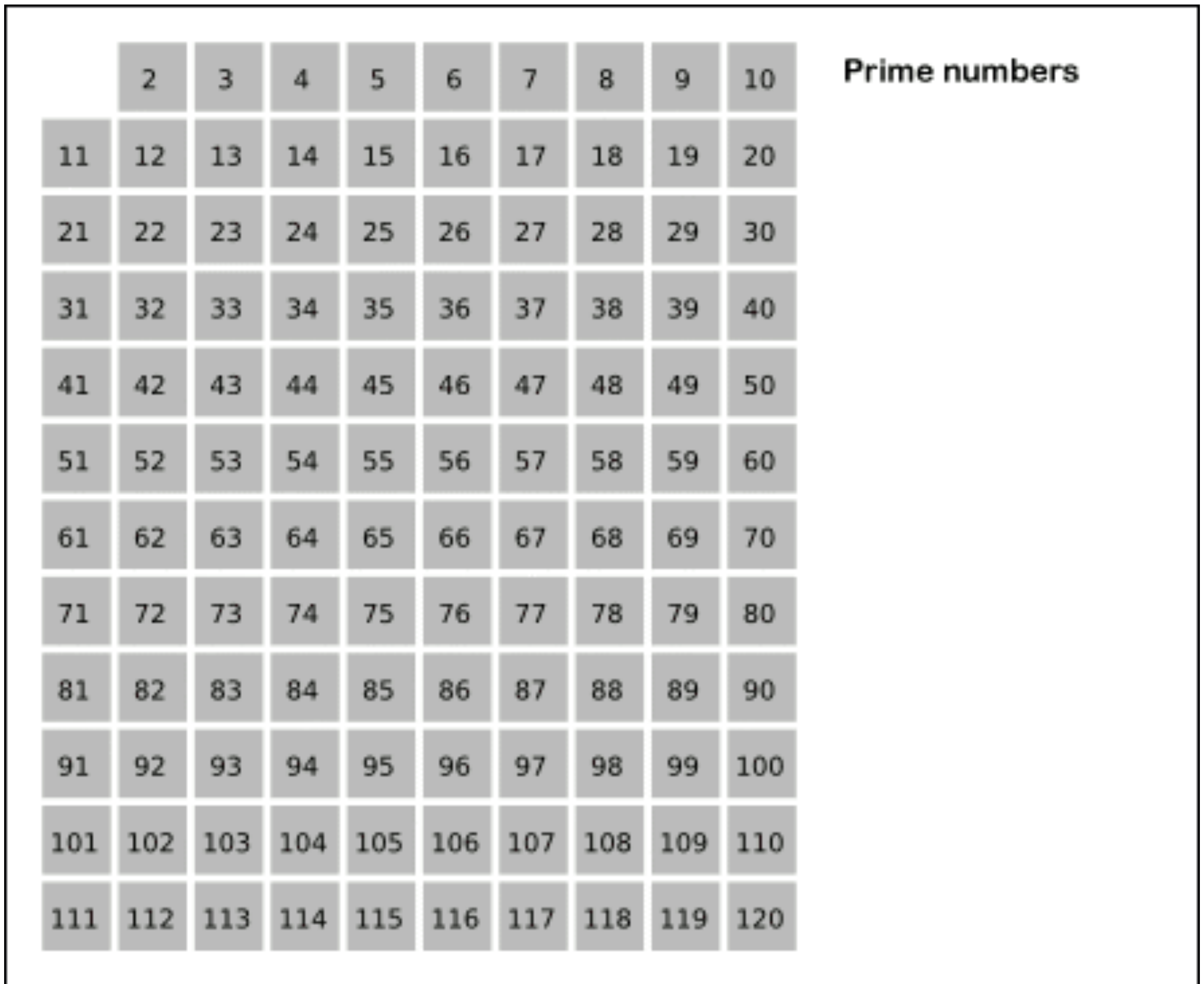
```
def isDivisible(m: Int, n: Int) = (m % n == 0)

val noSevens = nats withFilter (isDivisible(_, 7))
```

A Prime Sieve

```
def sieve(stream: Stream[Int]): Stream[Int] =
  ConsStream(stream.head,
             () => sieve(stream.tail withFilter
                           (x => !(isDivisible
                                     (x, stream.head)))))
```

Sieve of Eratosthenes



https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes

https://commons.wikimedia.org/w/index.php?curid=2810935

A Stream of Primes

```
val primes = sieve(intsFrom(2))
```

A Stream of Primes

```
> primes.head
res5: Int = 2
> primes.nth(1)
res6: Int = 3
> primes.nth(2)
res7: Int = 5
> primes.nth(3)
res8: Int = 7
```

Streams

```
def add(xs: Stream[Int],
        ys: Stream[Int]) :Stream[Int] = {

  (xs, ys) match {
    case (NilStream, _) => ys
    case (_, NilStream) => xs
    case (ConsStream(x,f), ConsStream(y,g)) =>
      ConsStream(x + y, () => add(f(), g()))
  }
}
```

Streams

```
def ones(): Stream[Int] = ConsStream(1, ones)
```

Alternative Definition of the Stream of Natural Numbers

```
def nats(): Stream[Int] =
  ConsStream(0, () => add(ones, nats))
```

Alternative Definition of the Fibonacci Stream

```
def fibs(): Stream[Int] =
  ConsStream(0,
             () => ConsStream(1,
                              () => add(fibs.tail, fibs)))
```

Powers of Two

```
def scaleStream(c: Int, stream: Stream[Int]): Stream[Int] =
  stream map (_ * c)

def powersOfTwo(): Stream[Int] =
  ConsStream(1, () => scaleStream(2, powersOfTwo))
```

Alternative Definition of the Stream of Primes

```
def primes() =
  ConsStream(2, () => intsFrom(3) withFilter isPrime)

def isPrime(n: Int): Boolean = {
  def sieve(next: Stream[Int]): Boolean = {
    if (square(next.head) > n) true
    else if (isDivisible(n, next.head)) false
    else sieve(next.tail)
  }
  sieve(primes)
}
```

Numeric Integration with Streams

$$S_i = c + \sum_{j=1}^{i} x_j dt$$

Numeric Integration with Streams

```
def integral(integrand: Stream[Double],
             init: Double,
             dt: Double) = {

  def inner(): Stream[Double] = {
    ConsStream(init,
               () => addStreams(scaleStream(dt,
                                             integrand),
                                inner))
  }
  inner
}
```

Streams and Local State

```
def withdraw(balance: Int, amounts: Stream[Int]):
Stream[Int] = {
  ConsStream(balance,
             () => withdraw(balance - amounts.head,
                            amounts.tail))
}
```

Discussion

- Our modeling of a bank account is a purely functional program without state
- Nevertheless:
  - If a user provides the stream of withdrawals, and
  - The stream of balances is displayed as outputs,
- The system will behave from a user's perspective as a stateful system

Discussion

- The key to understanding this paradox is that the “state” is in the world:
  - The user/bank system is stateful and provides the input stream
  - If we could “step outside” our own perspective in time, we could view our withdrawal stream as another stateless stream of transactions

Changing the State of Variables

Changing the State of Variables

- Thus far, we have focused solely on purely functional programs
- This approach has gotten us remarkably far
- Sometimes, it is difficult to structure a program without some notion of stateful variables:
  - I/O, GUIs
  - Modeling a stateful system in the world

Assignment and Local State

- We view the world as consisting of objects with state that changes over time
- It is often natural to model physical systems with computational objects with state that changes over time

Assignment and Local State

- If we choose to model the flow of time in the system by elapsed time in the computation, we need a way to change the state of objects as a program runs

- If we choose to model state using symbolic names in our program, we need an assignment operator to allow for changing the value associated with a name

Modeling an Address Book

```
class AddressBook() {
  val addresses: Map[String,String] = Map()

  def put(name: String, address: String) = {
    …
  }

  def lookup(name: String) = addresses(name)
}
```

Modeling an Address Book

```
class AddressBook() {
  var addresses: Map[String,String] = Map()

  def put(name: String, address: String) = {
    addresses = addresses + (name -> address)
  }

  def lookup(name: String) = addresses(name)
}
```

You now saw var; you are still not allowed to use it :)

Sameness and Change

- In the context of assignment, our notion of equality becomes far more complex

```
val petersAddressBook = new AddressBook()
val paulsAddressBook = new AddressBook()
```

```
val petersAddressBook = new AddressBook()
val paulsAddressBook = paulsAddressBook
```

Sameness and Change

- Effectively assignment forces us to view names as referring not to values, but to *places* that store values

Referential Transparency

- The notion that equals can be substituted for equals in an expression without changing the value of the expression is known as *referential transparency*
- Referential transparency is one of the distinguishing aspects of functional programming
- It is lost as soon as we introduce assignment

Referential Transparency

- Without referential transparency, the notion of what it means for two objects to be “the same” is far more difficult to explain
- One approach:
  - Modify one object and see whether the other object has changed in the same way

Referential Transparency

- One approach:
  - Modify one object and see whether the other object has changed in the same way
  - But that involves observing a single object twice
  - How do we know we are observing the same object both times?

Pitfalls of Imperative Programming

- The order of updates to variables is a classic source of bugs

```scala
def factorial(n: Int) = {
  var product = 1
  var counter = 1
  def iter(): Int = {
    if (counter > n) {
      product
    }
    else {
      product = product * counter
      counter = counter + 1
      iter()
    }
  }
  iter()
}
```

```scala
def factorial(n: Int) = {
  var product = 1
  var counter = 1
  def iter(): Int = {
    if (counter > n) {
      product
    }
    else {
      product = product * counter
      counter = counter + 1
      iter()
    }
  }
  iter()
}
```

What if the order of these updates were reversed?

Review: The Environment Model of Evaluation

- Environments map names to values
- Every expression is evaluated in the context of an environment

The Environment Model of Reduction

- To evaluate a name, simply reduce to the value it is mapped to in the environment

The Environment Model of Reduction

- To evaluate a function, reduce it to a *closure,* which consists of two parts:
  - The body of the function
  - The environment in which the body occurs

The Environment Model of Reduction

- Objects are also modeled as closures
    - What is the environment?
    - What corresponds to the body of the function?

The Environment Model of Reduction

- To evaluate an application of a closure
    - Extend the environment of the closure, mapping the function’s parameters to argument values
    - Evaluate the body of the closure in this new environment

Variable Rebinding in the Environment Model

- The environment model provides us with the necessary machinery to model stateful variables
- To evaluate a variable *v* assignment:
  - Rebind the value *v* maps to in the environment in which the assignment occurs

Rebinding a Variable in an Environment

- The rebound value of *v* is then used in all subsequent reductions involving the same environment
    - Includes closures involving that environment
- This model of variable assignment pushes the notion of state out to environments
- The “places” referred to by variables are simply components of environments

Example: Pseudo-Random Number Generation

- There are many approaches to generating a pseudo-random stream of `Int` values

- One common approach is to define a *linear congruential generator (LCG)*:

$$X_{n+1} = (aX_n + c) \bmod m$$

- The pseudo-random numbers are the elements of this recurrence

Linear Congruential Generators

- LCGs can produce generators capable of passing formal tests for randomness
- The quality of the results is highly dependent on the initial values selected
- Poor statistical properties
- Not well suited for cryptographic purposes

A Linear Congruent Generator (C++11 `minstd_rand`)

```
def makeRandomGenerator(): () => Int = {
  val a = 48271
  val b = 0
  val m = Int.MaxValue
  var seed = 2

  def inner() = {
    seed = (a*seed + b) % m
    seed
  }
  inner
}
```

A Linear Congruent Generator (`C++11 minstd_rand`)

```
val g = makeRandomGenerator()<E> ↦
val g =
< def inner() = {
        seed = (a*seed + b) % m
        seed
  } ,
  val a = 48271
  val b = 0
  val m = Int.MaxValue
  var seed = 2 >
```

```
g()<E> ↦
< def inner() = {
        seed = (a*seed + b) % m
        seed
  } ,
  val a = 48271
  val b = 0
  val m = Int.MaxValue
  var seed = 2 >()<E> ↦
```

```
seed = (a*seed + b) % m
seed,
< val a = 48271
  val b = 0
  val m = Int.MaxValue
  var seed = 2 >
↦
seed = (48271*2 + 0) % Int.MaxValue
seed,
< val a = 48271
  val b = 0
  val m = Int.MaxValue
  var seed = 2 >
↦
```

```
seed, <val a = 48271
       val b = 0
       val m = Int.MaxValue
       var seed = 96542>
```

$\mapsto$

96542

```
seed, <val a = 48271
       val b = 0
       val m = Int.MaxValue
       var seed = 96542>
```

↦

```
96542
```

And now the environment closing over generator g binds seed to 96542.