Comp 311
Functional Programming

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Passing Function Literals As Arguments

```
val xs = Cons(1,Cons(2,Cons(3,Cons(4,Cons(5,Cons(6,Empty))))))

filter(_ < 3, xs) ↦* Cons(1,Cons(2,Empty))
```

Guidelines On Using Function Literals

- Function literals are well-suited to situations in which:
  - The function is only used once
  - The function is not recursive
  - The function does not constitute a key concept in the problem domain

Comprehensions

$$\{2x \mid x \in xs\}$$

Mapping a Computation Over a List

```
def double(xs: List) = {
  xs match {
    case Empty => Empty
    case Cons(y,ys) => Cons(y * y, double(ys))
  }
}
```

Mapping a Computation Over a List

```
def negate(xs: List) = {
  xs match {
    case Empty => Empty
    case Cons(y,ys) => (-y, negate(ys))
  }
}
```

Negation as a Comprehension

$$\{-x \mid x \in xs\}$$

Generalizing a Mapping Computation

```
def map(f: Int => Int, xs: List) = {
  xs match {
    case Empty => Empty
    case Cons(y,ys) => Cons(f(y), map(f,ys))
  }
}
```

Mapping a Computation Over a List

```
val xs = Cons(1,Cons(2,Cons(3,Cons(4,Cons(5,Cons(6,Empty))))))

negate(xs) ↦*
Cons(-1,Cons(-2,Cons(-3,Cons(-4,Cons(-5,Cons(-6,Empty))))))

double(xs) ↦*
Cons(1,Cons(4,Cons(9,Cons(16,Cons(25,Cons(36,Empty))))))
```

Mapping a Computation Over a List

```
val xs = Cons(1,Cons(2,Cons(3,Cons(4,Cons(5,Cons(6,Empty))))))

map(-_, xs) ↦*
Cons(-1,Cons(-2,Cons(-3,Cons(-4,Cons(-5,Cons(-6,Empty))))))

map(x => x * x, xs) ↦*
Cons(1,Cons(4,Cons(9,Cons(16,Cons(25,Cons(36,Empty))))))
```

Recall Our Sum Function Over Lists

```
def sum(xs: List): Int = {
  xs match {
    case Empty => 0
    case Cons(y,ys) => y + sum(ys)
  }
}
```

In Mathematics, We Might Write this as a Summation

$$\sum_{x \in xs} x$$

And Our Product Function Over Lists

```
def product(xs: List): Int = {
  xs match {
    case Empty => 1
    case Cons(y,ys) => y * sum(ys)
  }
}
```

In Mathematics, We Might Write this as a Product

$$\prod_{x \in xs} x$$

We Abstract to a Reduction Function Over Lists

```
def reduce(base: Int, f: (Int, Int) => Int, xs: List): Int = {
  xs match {
    case Empty => base
    case Cons(y,ys) => f(y, reduce(base, f, ys))
  }
}
```

Example Reductions

```
val xs = Cons(1,Cons(2,Cons(3,Cons(4,Cons(5,Cons(6,Empty))))))

reduce(0, (x,y) => x + y, xs) ↦* 21

reduce(1, (x,y) => x * y, xs) ↦* 720
```

Min and Max

```
def max(xs: List) = {
  reduce(Int.MinValue, (x,y) => if (x > y) x else y, xs)
}

def min(xs: List) = {
  reduce(Int.MaxValue, (x,y) => if (x < y) x else y, xs)
}
```

Simplifying Function Literals

- When *each* parameter is used only once in the body of a function literal, and in the order in which they are passed:
  - We can drop the parameter list
  - We simply write the body with an _ at the place where each parameter is used

For example,

```
((x: Int, y: Int) => (x + y))
```

becomes

```
_ + _
```

Example Reductions

```
val xs = Cons(1,Cons(2,Cons(3,Cons(4,Cons(5,Cons(6,Empty))))))

reduce(0, _+_, xs) ↦* 21

reduce(1, _*_, xs) ↦* 720
```

Note the multiple parameters

Combinations of Maps and Reductions

$$\sum_{x \in xs} x^2 + 1$$

Combinations of Maps and Reductions

```
reduce(0, _+_, map(x => x*x + 1, xs))
```

Summation

```
def summation(xs: List, f: Int => Int) =
   reduce(0, _+_, map(f, xs))
```

Summation

```
def square(x:Int) = x * x

summation(xs, square(_)+1)
```

More Syntactic Sugar

- Functions defined with `def` can be passed as arguments whenever an expression of a compatible function type is expected

- What constitutes a compatible function type?

Partially Applied Functions

- If we want to pass a function as an argument, but supply some of the arguments to the function ourselves, we can wrap an application to the function in a function literal:

```
map(x => x + 1, xs)
```

Partially Applied Functions

- If we want to pass a function as an argument, but supply some of the arguments to the function ourselves, we can wrap an application to the function in a function literal:

```
map(x => x + 1, xs)
```

which is equivalent to

```
map(_ + 1, xs)
```

Partially Applied Functions

- **Eta Expansion:** Wrapping a function in function literal that takes all of the arguments of f and immediately calls f with those arguments

```
(x:Int) => square(x)
```

is equivalent to

```
square
```

Mapping a Computation Over a List

We can use eta expansion to pass operators as arguments:

```
map(x => -x, xs)
```

Mapping a Computation Over a List

We can use eta expansion to pass operators as arguments:

```
map(-_, xs)
```

Returning Functions as Values

We Can Define Functions That Return Other Functions as Values

```
def add(x: Int): Int => Int = {
  def addX(y: Int) = x + y
  addX
}
```

We Can Define Functions That Return Other Functions as Values

```
def add(x: Int): Int => Int = {
  def addX(y: Int) = x + y
  addX
}
```

The explicit return type is needed because Scala type inference assumes an unapplied function is an error

We Can Define Functions That Return Other Functions as Values

```
def add(x: Int) = {
  def addX(y: Int) = x + y
  addX _
}
```

Alternatively, we can eta-expand addX to assure the type checker that we really do intend to return a function

We Can Define Functions That Return Other Functions as Values

```
def add(x: Int) = {
  def addX(y: Int) = x + y
  addX _
}
```

An underscore outside of parentheses in a function application denotes the entire tuple of arguments passed to the function

We Can Define Functions That Return Other Functions as Values

```
def add(x: Int) = x + (_: Int)
```

We can instead define add by *partially* eta-expanding the + operator. But then we need to annotate the second operand with a type.

Aside: Type Annotations

- In general, an expression annotated with a type is itself an expression:

```
expr: Type
```

- If the static type of `expr` is a subtype of `Type`, then the type of `expr:Type` is `Type`

Partial Eta-Expansion

- We can partially eta-expand any function, but we need to annotate the argument types:

```
def reduce0 =
    reduce(0, _: (Int, Int) => Int, _: List)
```

Derivatives

```
def derivative(f: Double => Double, dx: Double) =
  (x: Double) =>
    (f(x + dx) - f(x)) /
      dx
```

Derivatives

```
def f(x: Double) = x * x
def Df = derivative(f, 0.00001)

f(4) ↦ 16
Df(4) ↦ 8.00000999952033
```

Encapsulating dx

```
def D(f: Double => Double) = {
  val dx = 0.00001
  (x: Double) =>
    (f(x + dx) - f(x)) /
      dx
}
```

Encapsulating dx

```
def D(f: Double => Double) = {
  val dx = 0.00001
  (x: Double) =>
    (f(x + dx) - f(x)) /
      dx
}
```

Our returned function "remembers" these values

Applying a Derivative

```
def D(f: Double => Double) = {
  val dx = 0.00001
  (x: Double) =>
    (f(x + dx) - f(x)) /
      dx
}
```

```
D(f)(4) ↦

D((x: Double) => x * x))(4) ↦
```

Applying a Derivative

```
D((x: Double) => x * x))(4) ↦

{val dx = 0.00001
 (x: Double) =>
    ((x: Double) => x * x)(x + dx) -
     (x: Double) => x * x)(x)) /
        dx     }(4) ↦
```

Applying a Derivative

```
{(x: Double) =>
    ((x: Double) => x * x)(x + 0.00001) -
     (x: Double) => x * x)(x)) /
       0.00001}(4) ↦

((x: Double) => x * x)(4 + 0.00001) -
  (x: Double) => x * x)(4)) /
 0.00001 ↦
```

We must be careful to substitute only corresponding occurrences of x

Applying a Derivative

```
((x: Double) => x * x)(4 + 0.00001) -
  (x: Double) => x * x)(4)) /
 0.00001 ↦

((x: Double) => x * x)(4.00001) -
  (x: Double) => x * x)(4)) /
 0.00001 ↦

((4.00001 * 4.00001) - (4 * 4)) /
 0.00001 ↦
```

Applying a Derivative

((4.00001 * 4.00001) - (4 * 4)) /
0.00001 ↦

(16.000080000099995 - 16) /
0.00001 ↦

8.000000999952033E-5 / 0.00001 ↦

8.000000999952033

Safe Substitution

Applying a Derivative

```
{(x: Double) =>
    ((x: Double) => x * x)(x + 0.00001) -
     (x: Double) => x * x)(x)) /
       0.00001}(4) ↦

((x: Double) => x * x)(4 + 0.00001) -
  (x: Double) => x * x)(4)) /
 0.00001
```

In cases like this one, we can avoid accidental variable capture by selective renaming

Safe Substitution
(a.k.a. Alpha Renaming)

- We can ensure we never accidentally substitute the wrong parameters by automatically renaming constants, functions, and parameters with *fresh* names
  - A fresh name must not capture a name referred to in the scope of a parameter
  - A fresh name must not be captured by a name in an enclosing scope

Applying a Derivative

```
{(x: Double) =>
    ((y: Double) => y * y)(x + 0.00001) -
     (z: Double) => z * z)(x)) /
       0.00001}(4) ↦

((y: Double) => y * y)(4 + 0.00001) -
  (z: Double) => z * z)(4)) /
 0.00001
```

Function Equivalence

- Now we have seen the three forms of function equivalence stipulated by the Lambda Calculus:
  - Alpha Renaming: Changing the names of a function’s parameters does not affect the meaning of the function
  - Beta Reduction: To apply a function to an argument, reduce to the body of the function, substituting occurrences of the parameter with the corresponding argument
  - Eta Equivalence: Two functions are equivalent iff they are *extensionally equivalent:* They give the same results for all arguments

Parametric Types

Parametric Types

- We have defined two forms of lists: lists of ints and lists of shapes
- Many computations useful for one are useful for the other:
  - Map, reduce, filter, etc.
- It would be better to define lists and their operations once for all of these cases

Parametric Types

- Higher-order functions take functions as arguments and return functions as results
- Likewise, *parametric types,* a.k.a., a *generic types*, takes types as arguments and return types as results

Parametric Lists

- Every application of this parametric type to an argument yields a new type:

```
abstract class List[T] {
  def ++(ys: List[T]): List[T]
}
```

Parametric Lists

- Every application of this parametric type to an argument yields a new type:

```
abstract class List[T <: Any] {
  def ++(ys: List[T]): List[T]
}
```

- We augment the declarations of type parameters to permit an upper bound on all instantiations of a parameter
  - By default, the bound is `Any`

Syntax of Parametric Class Definitions

```
<modifiers> class C[T1 <: N,..,TN <: N] extends N {
   <ordinary class body>
}
```

- We denote “naked” type parameters as `T1, T2,` etc.
- We denote all other types with `N, M`, etc.

Syntax of Parametric Class Definitions

```
<modifiers> class C[T1 <: N,..,TN <: N] extends N {
   <ordinary class body>
}
```

- Declared type parameters T1, …, TN are in scope throughout the entire class definition, including:
  - The bounds of type parameters
  - The `extends` clause
- Object definitions must not be parametric

Parametric Lists

- Every application of this parametric type yields a new type:

```
List[Int]
List[String]
List[List[Double]]
etc.
```

Parametric Lists

- Every application (a.k.a., *instantiation*) of this parametric type yields a new type:

```
abstract class List[T] {
  def ++(ys: List[T]): List[T]
}
```

Note that our parametric type can be instantiated with type parameters, including its own!

Parametric Lists

```scala
case class Empty[S]() extends List[S] {
  def ++(ys: List[S]) = ys
}

case class Cons[T](head: T, tail: List[T]) extends List[T] {
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)
}
```

Parametric Lists

```
case class Empty[S]() extends List[S] {
  def ++(ys: List[S]) = ys
}

case class Cons[T](head: T, tail: List[T]) extends List[T] {
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)
}
```

Our definition requires a separate type Empty[S] for every instantiation of S. Thus we must define Empty as a class rather than an object.

Type Environments

- To explain how to type check expressions in the context of parametric types, we must introduce the notion of *environments*

- We define a type parameter environment to hold a collection of zero or more type parameter declarations with their bounds

- Type environments can be extended with more declarations

Type Checking a Class Definition

- To type check a parametric class definition:
    - Check the declarations of the class in a new type parameter environment that extends the enclosing environment with all its type parameters

Type Checking a Function Definition

- To type check a function definition in environment E:
  - Check that the types of all parameters are *well-formed*
  - Find the type of the body of the function, substituting occurrences of parameters with their types
  - Ensure that the type of the body is a subtype of the declared return type (in environment E)

Well-Formedness of Types

- A type is well-formed in environment E iff:
  - If it is a well-defined non-parametric type
  - It is a type parameter T in environment E
  - It is an instantiation of a defined parametric type and:
    - All of its type arguments are well-formed types in E
    - All of its type arguments respect the bounds on their corresponding type parameters

Subtyping With Environments

- It is non-sensical to compare types in separate type environments:

```
case class Empty[S]() extends List[S] {
  def ++(ys: List[S]) = ys
}

case class Cons[T](head: T, tail: List[T]) extends List[T] {
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)
}
```

- Is S a subtype of T?

Subtyping With Environments

- We must modify our subtyping rules to refer to an environment E:
  - S <: S in E
  - If S <: T in E and T <: U in E then S <: U in E

Subtyping With Environments

- If:
  - `class C[T1,..,TN] extends D[U1,…UM]`
  - and `X1,…,XN` are well-formed in `E`
  - then `C[X1,…XN] <: D[U1,…,UM][T1↦X1,…,TN↦XN]` in `E`

Subtyping With Environments

- If:
  - `class C[T1,..,TN] extends D[U1,…UM]`
  - and `X1,…,XN` are well-formed in `E`
  - then `C[X1,…XN] <: D[U1,…,UM][T1↦X1,…,TN↦XN]` in `E`

We use this notation to indicate safe substitution of `T1` for `X1`, … `TN` for `XN` in `D[U1,…,UM]`

Covariance

- Can one instantiation of a parametric type be a subtype of another?

- Currently our rules allow this only in the reflexive case:

```
List[Int] <: List[Int] in E
```

Covariance

- It would be useful to allow some instantiations to be subtypes of another
- For example, we would like it to be the case that:

```
List[Int] <: List[Any]
```

Covariance

- In general, we say that a parametric type `C` is covariant with respect to its type parameter `S` if:

```
S <: T in E
```

implies

```
C[S] <: C[T] in E
```

- We must be careful that such relationships do not break the soundness of our type system

Covariance

- For a parametric type such as:

```
abstract class List[T <: Any] {
  def ++(ys: List[T]): List[T]
}
```

- And types `S` and `T`, such that `S <: T` in some environment `E`:
  - What must we check about the body of class `List` to allow for `List[S] <: List[T]` in `E`?

Covariance

- Consider instantiations for types `String` and `Any`:

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] {
  def ++(ys: List[String]): List[String]
}
```

Covariance

- If these were ordinary classes connected by an `extends` class:
  - We would need to ensure that the overriding definition of `++` in class `List[String]` was compatible with the overridden definition in `List[Any]`

Covariance

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] extends List[Any] {
  def ++(ys: List[String]): List[String]
}
```

Covariance

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] extends List[Any] {
  def ++(ys: List[String]): List[String]
}
```

But if List[String] <: List[Any] in E
then this is not a valid override

Covariance

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] extends List[Any] {
  def ++(ys: List[String]): List[String]
}
```

On the other hand, the return types are not problematic

Covariance

- From our example, we can glean the following rule:
    - We allow a parametric class `C` to be covariant with respect to a type parameter `T` so long as `T` does not appear in the types of the method parameters of `C`

Covariance

```
abstract class List[+T] {}
```

- We stipulate that a parametric type is covariant in a parameter `T` by prefixing a `+` at the definition of `T`
- (We will return to our definition of `append` later)

Covariance

```
case object Empty extends List[Nothing] {
}

case class Cons[+T](head: T, tail: List[T])
extends List[T] {
}
```