Comp 311
Functional Programming

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Announcements

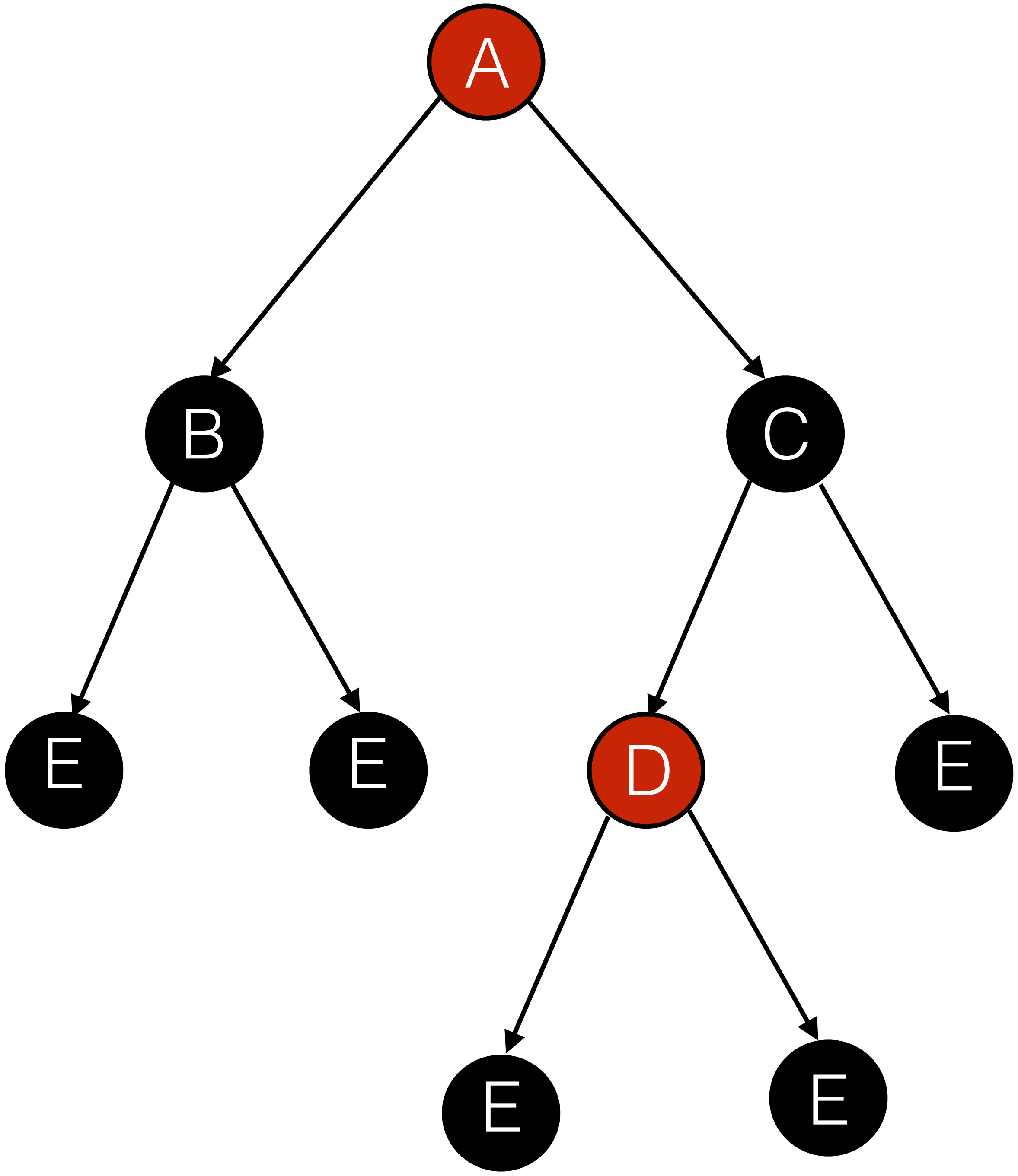
- Guest Lecture on Tuesday:
    - Shams Imam: Co-routines in Scala
- My office hours next week will be at Thursday 4-5pm

Red-Black Trees Continued

Review: Red-Black Trees

- Every node is colored either red or black
- All leaf nodes are black
- No red node has a red child
- Every path from the root to a leaf contains the same number of black nodes

Review: An Example Red-Black Tree



Review: Strategy for Insertion

- We insert new elements as usual, but then rebalance the tree to maintain the red-black invariants

- At the end of the rebalancing, we recolor the root to black

    - This cannot violate our invariants

Red-Black Trees

```
abstract class Tree[T <: Ordered[T]] {
  def empty = Leaf[T]
  def contains(x: T): Boolean
  def insert(x: T): Tree[T] = insertChildren(x) match {
    case Branch(c,l,e,r) => Branch(Black, l, e, r)
  }
  def insertChildren(x: T): Branch[T]
}
```

We call a helper function insertChildren,
which performs the insertion and rebalancing.

Red-Black Trees

```
abstract class Tree[T <: Ordered[T]] {
  def empty = Leaf[T]
  def contains(x: T): Boolean
  def insert(x: T): Tree[T] = insertChildren(x) match {
    case Branch(c,l,e,r) => Branch(Black, l, e, r)
  }
  def insertChildren(x: T): Branch[T]
}
```

We take the result from insertChildren, ignore the color of the root and return a tree that is nearly identical except that the root is colored black.

Red-Black Trees

```scala
case class Leaf[T <: Ordered[T]]() extends Tree[T] {
  def contains(x: T) = false
  def insertChildren(x: T) = Branch(Red, this, x, this)
}
```

Red-Black Trees

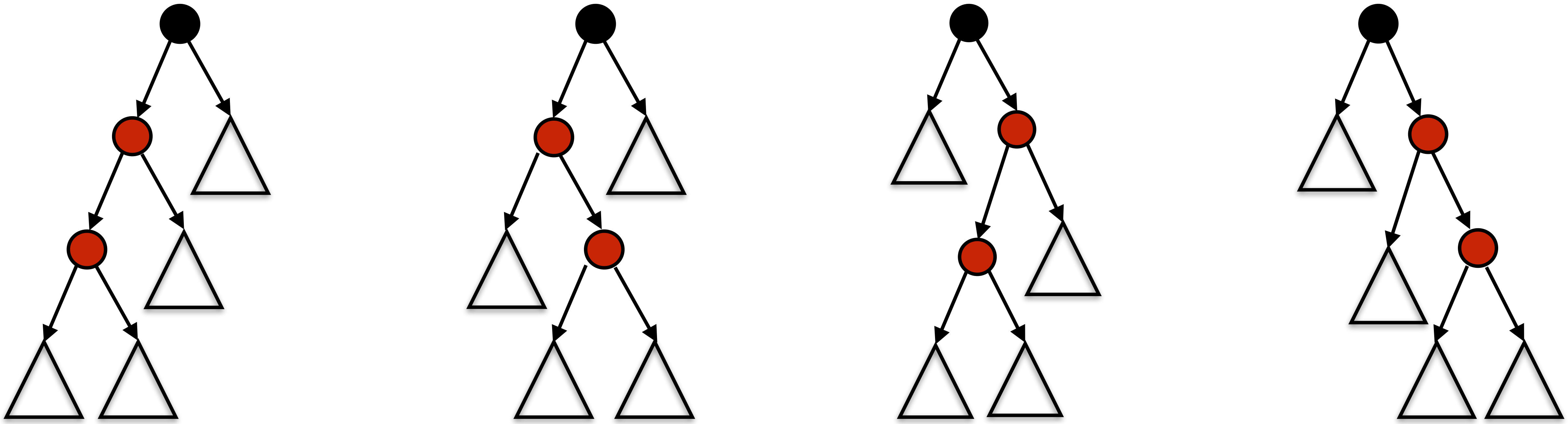
```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {

  def contains(x: T) = {
    if (x < element) left contains x
    else if (x > element) right contains x
    else true // x == element
  }
  …
}
```

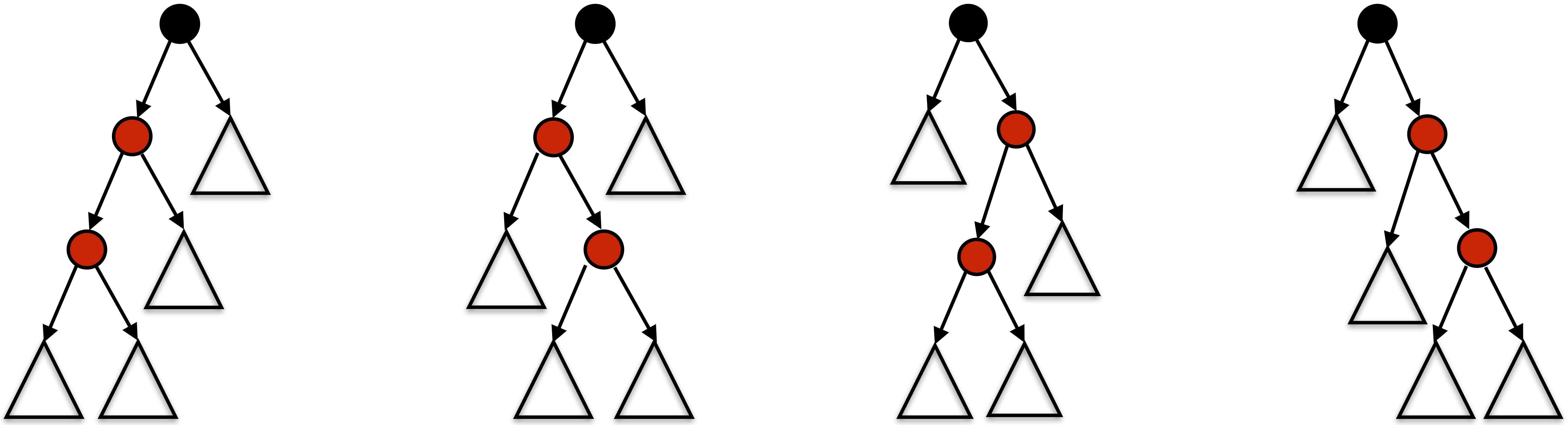
Red-Black Trees

```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {
  …
  def insertChildren(x: T) = {
    if (x < element)
      balance(color, left insertChildren x, element, right)
    else if (x > element)
      balance(color, left, element, right insertChildren x)
    else this
  }
  …
}
```

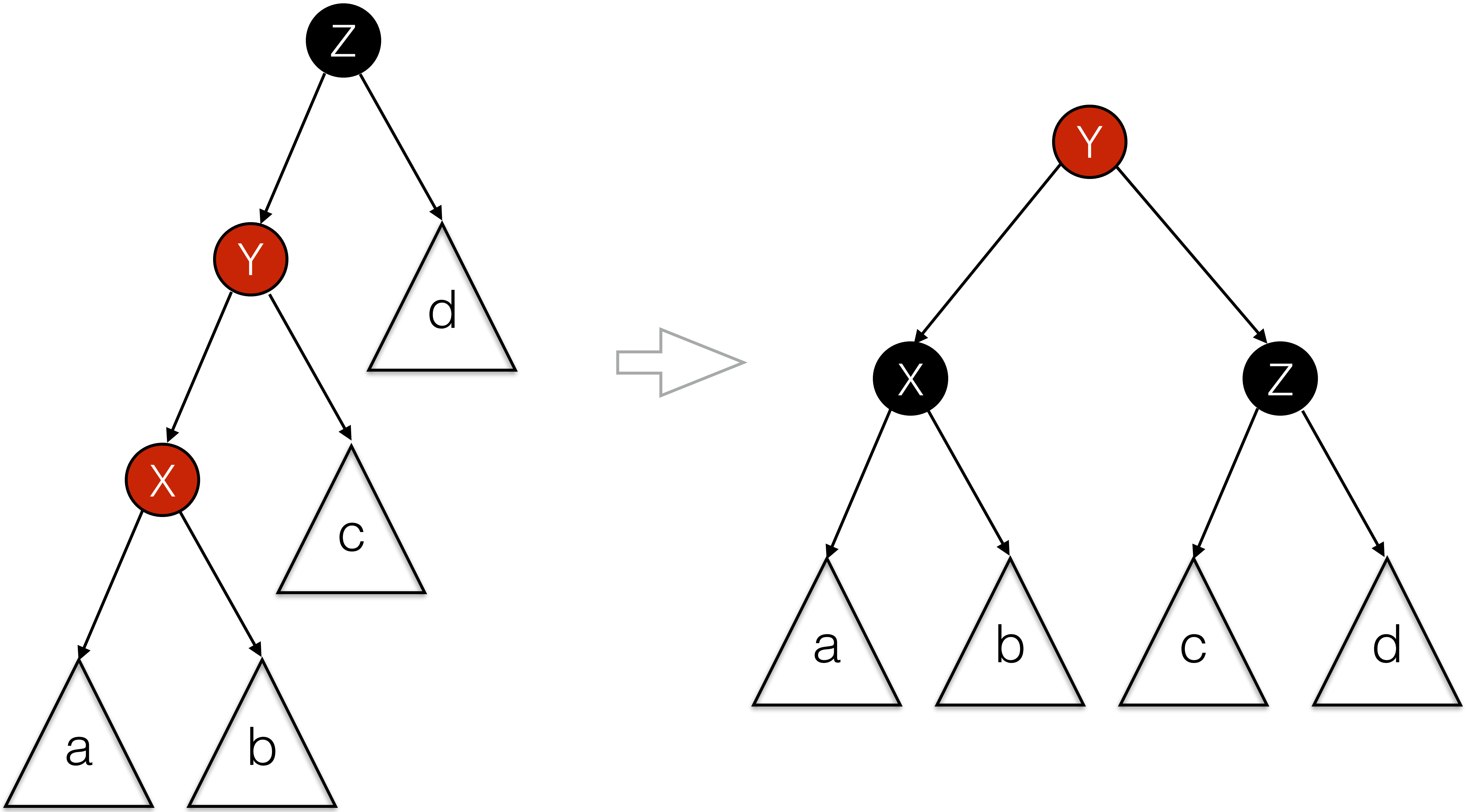
Rebalancing:
There are Four Cases to Consider



Rebalancing:
There are Four Cases to Consider



We use pattern matching to enumerate the cases.


Z
Y
d
X
c
a
b
Y
X
Z
a
b
c
d

```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {





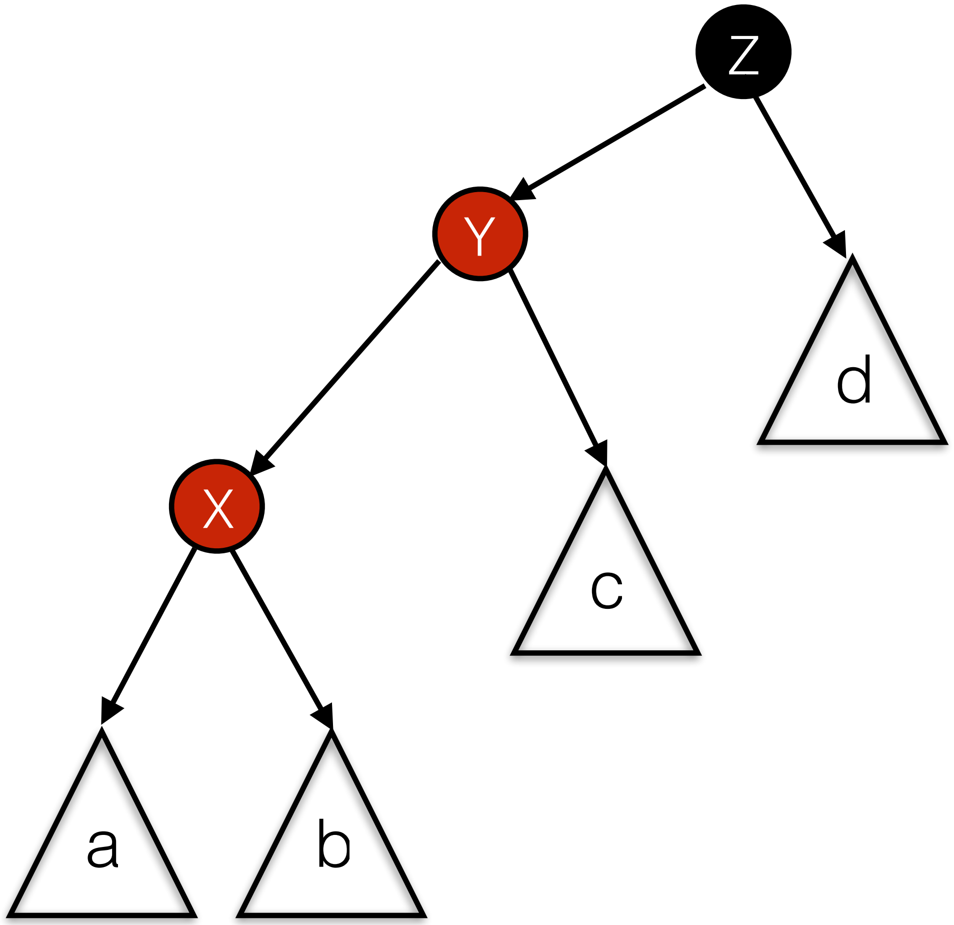






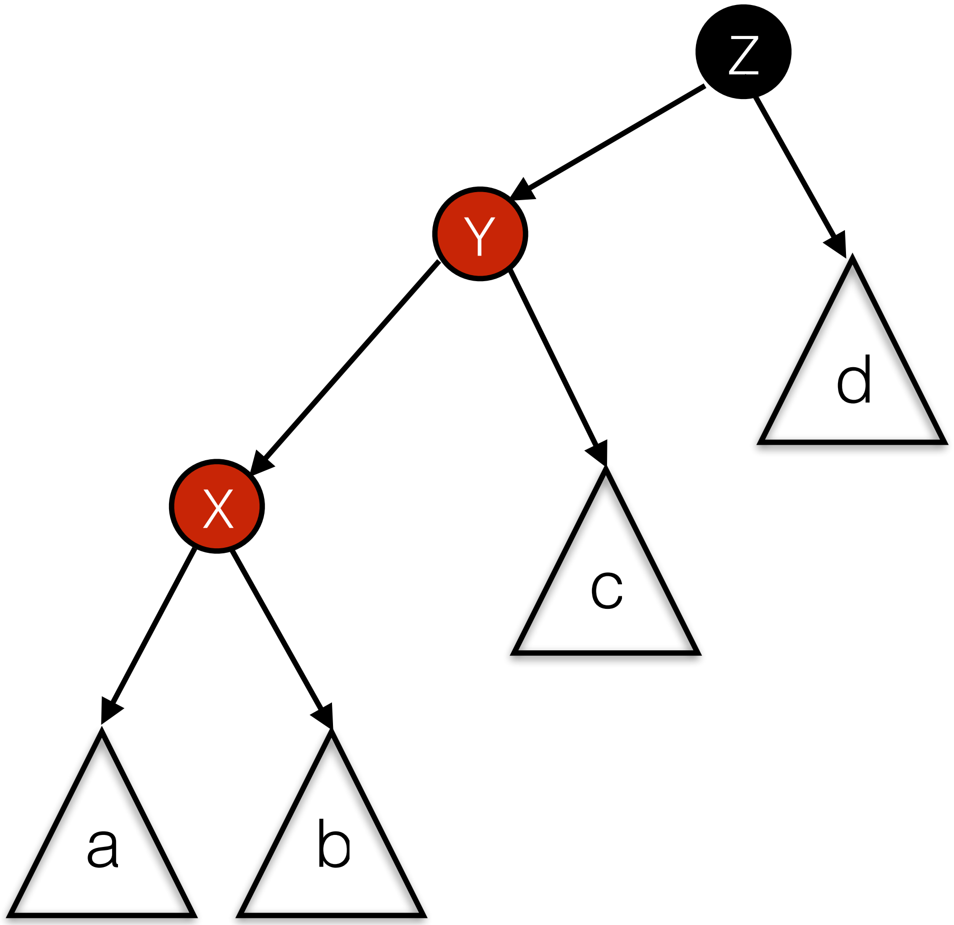

      …
    }
  }
  …
```

```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
```



```
        …
    }
  }
  …
```

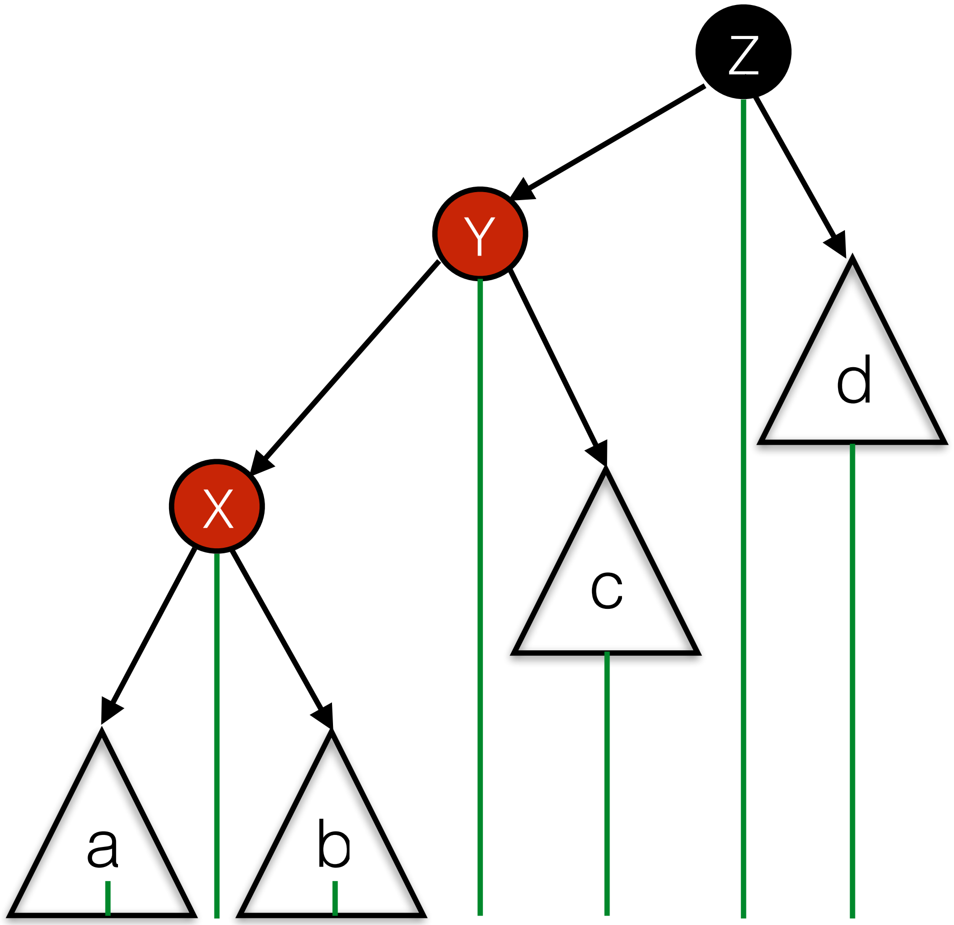
```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
```



```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>

      …
    }
  }
  …
```

```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
```



```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>

      …
    }
  }
  …
```

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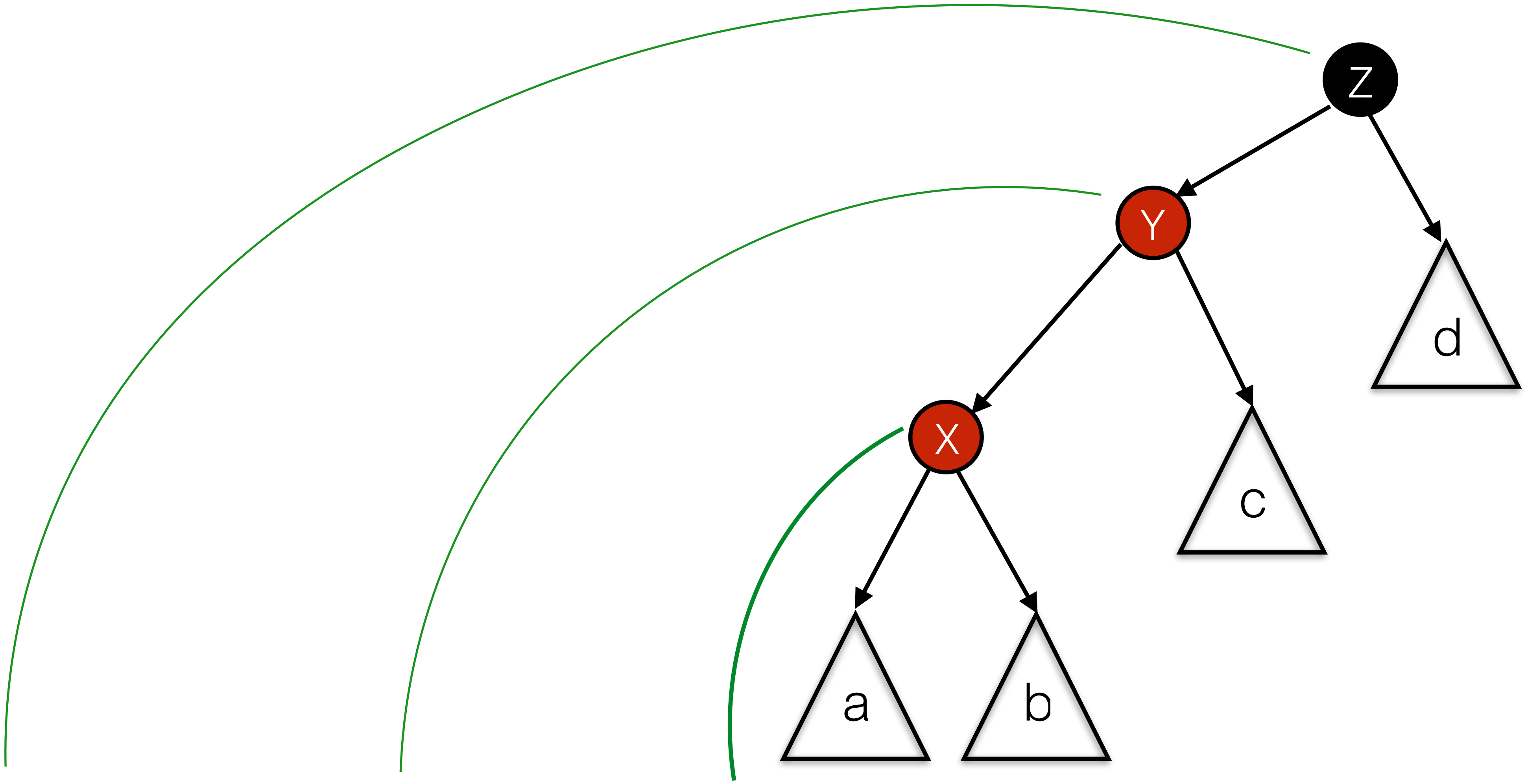

```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>

      …
    }
  }
  …
```

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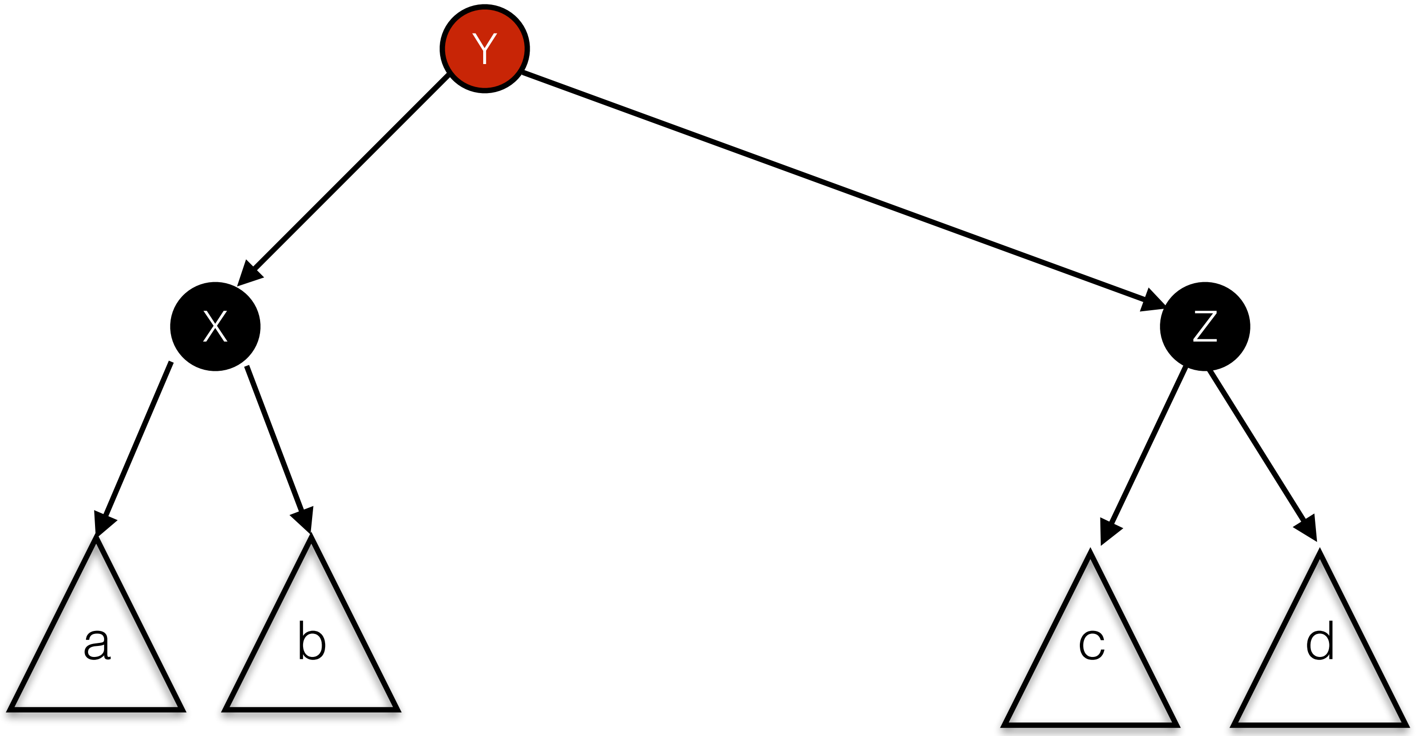
```



```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>

      …
    }
  }
  …
```

```
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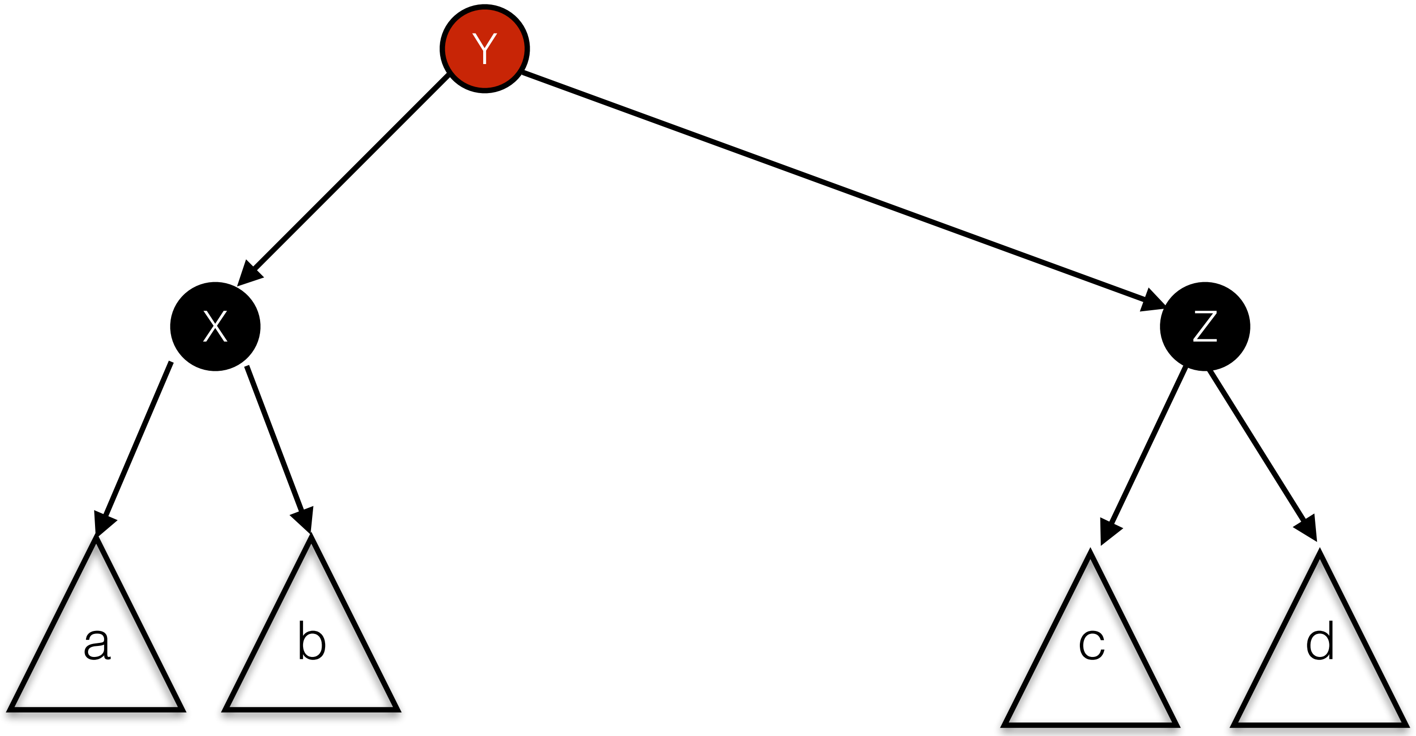
```



```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      …
    }
  }
  …
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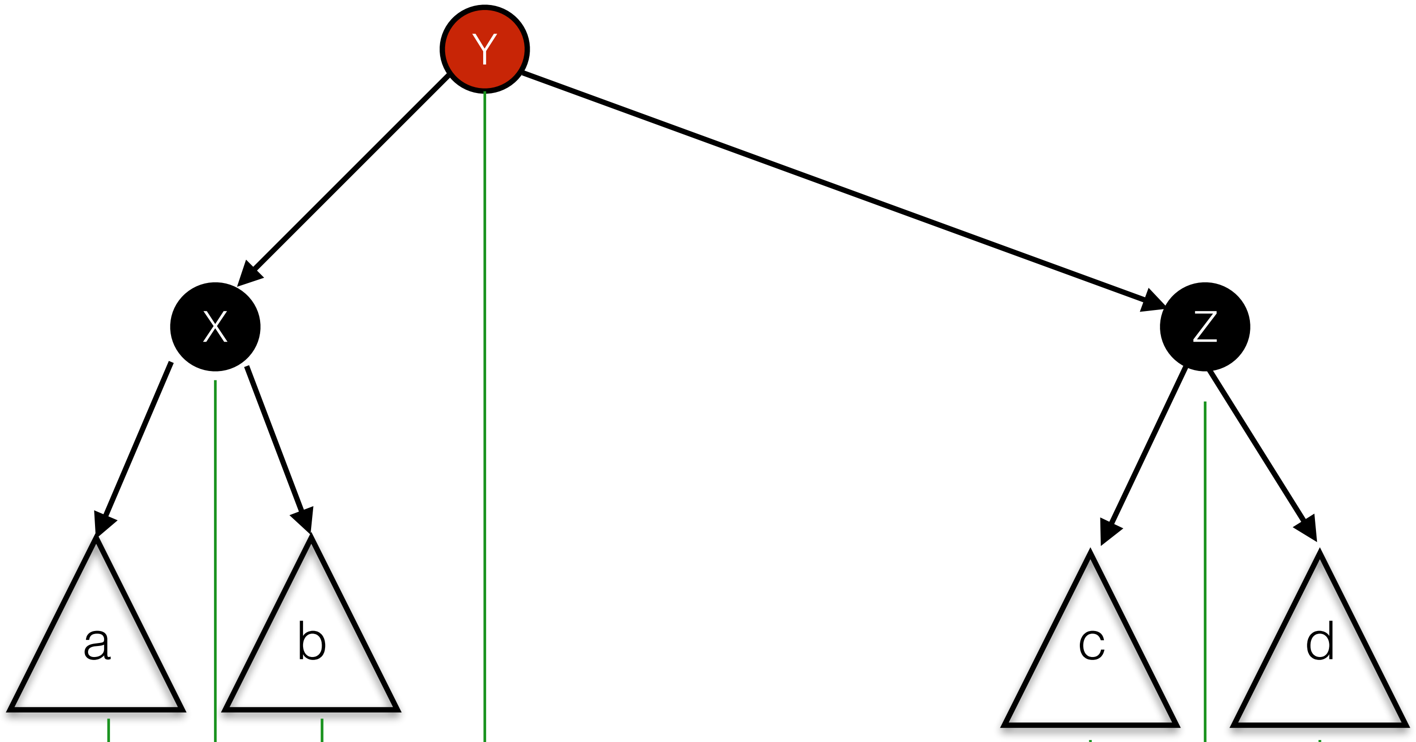

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        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      …
    }
  }
  …
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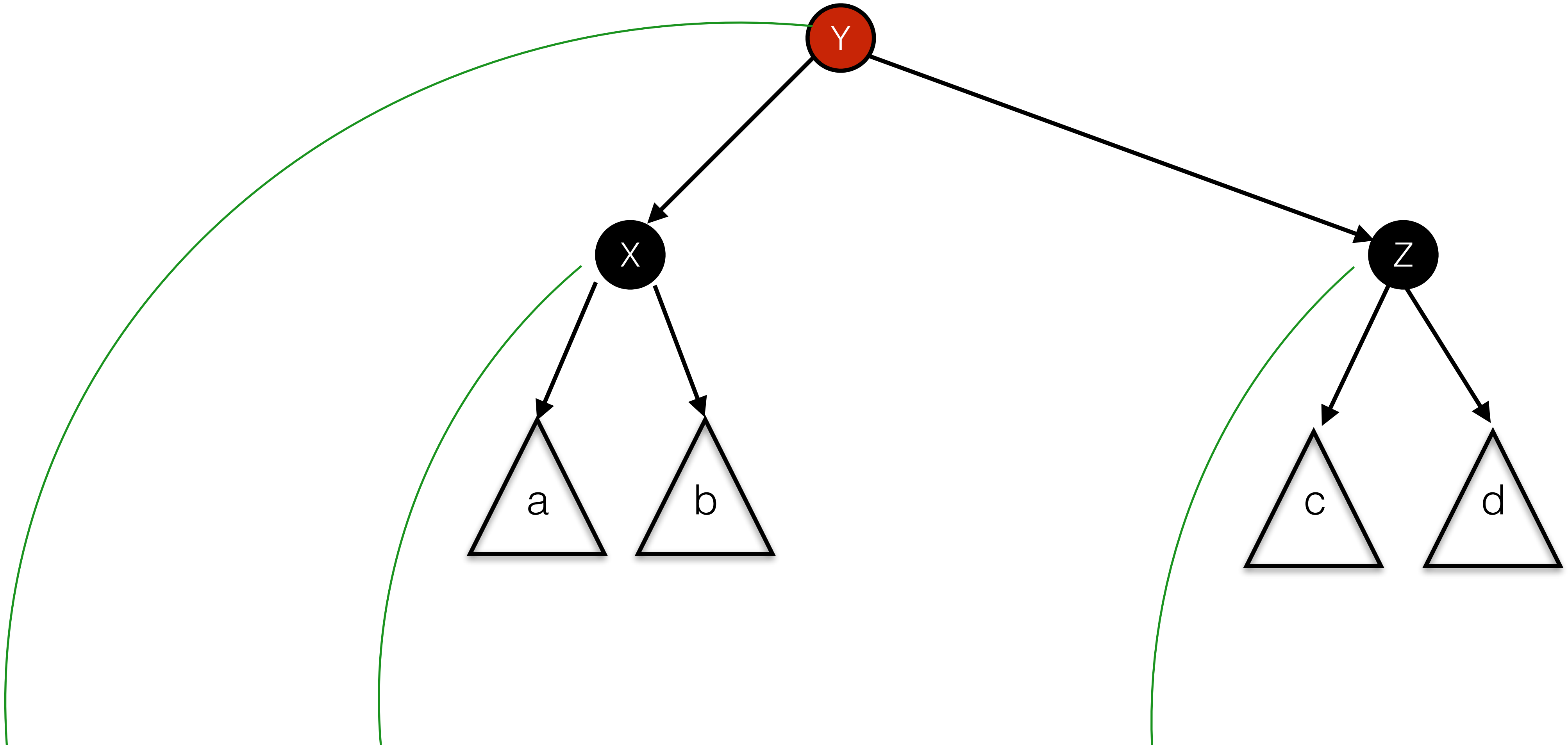

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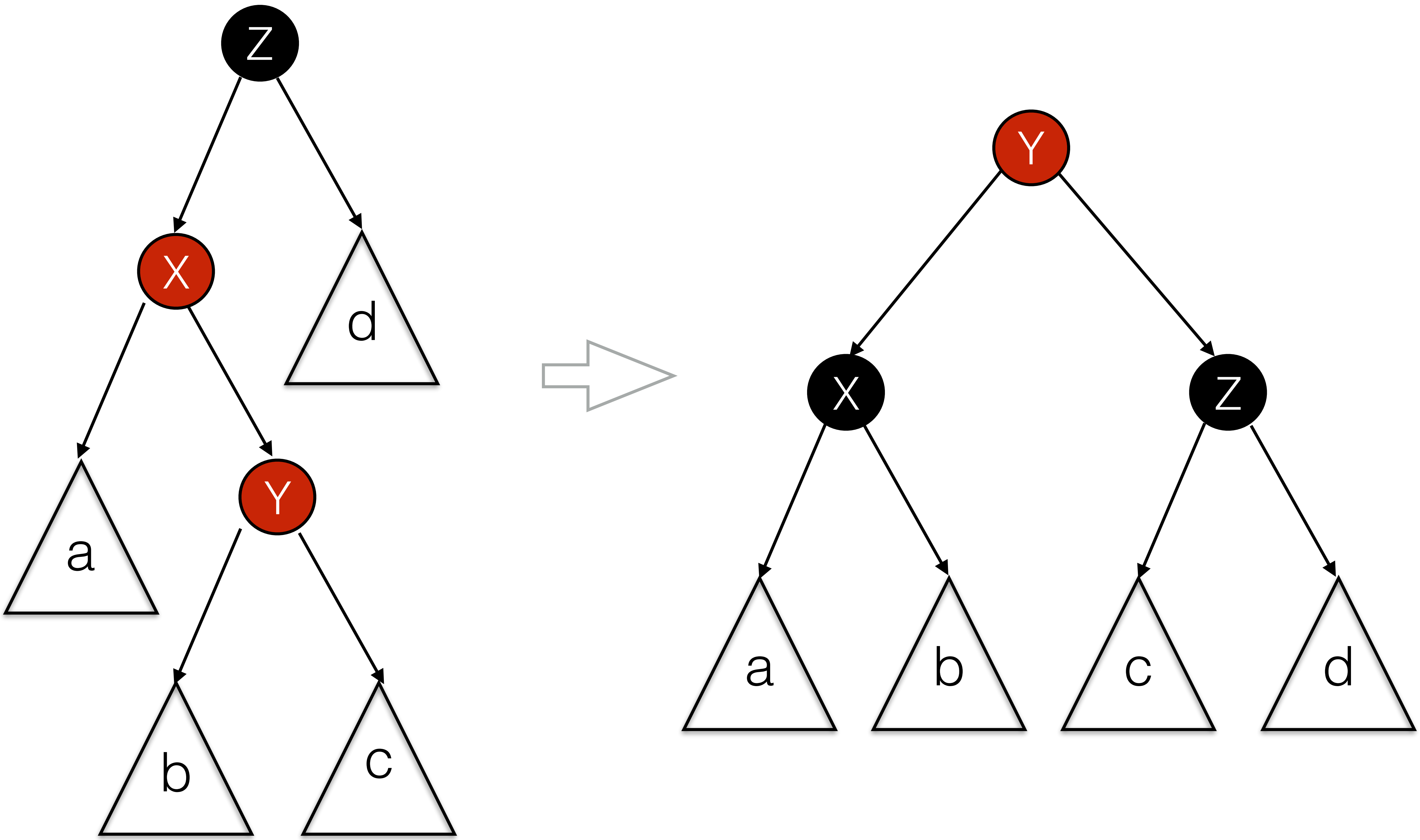
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      …
    }
  }
  …
```


Z
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c
Y
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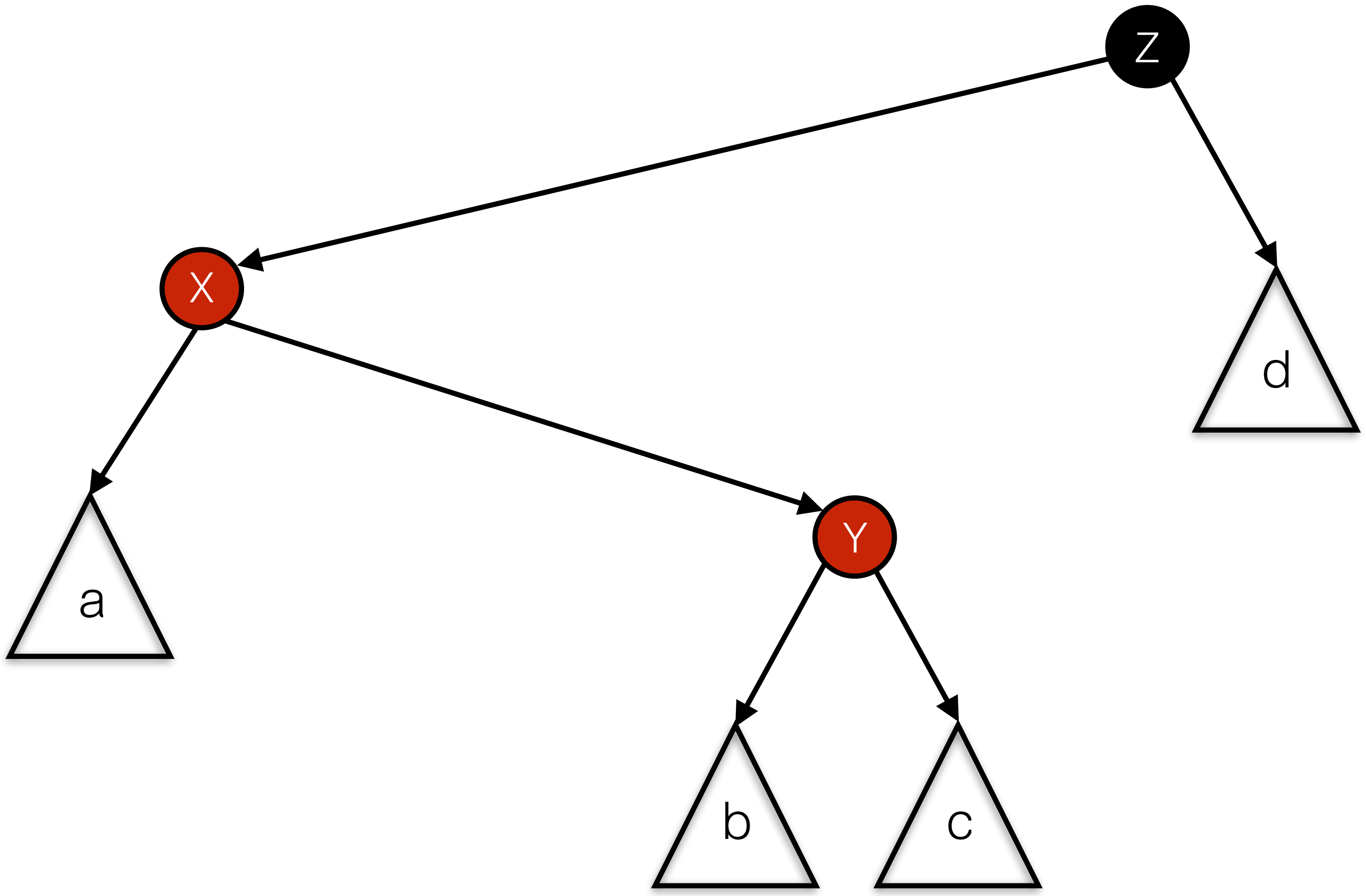

```
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>


      …
    }
  }
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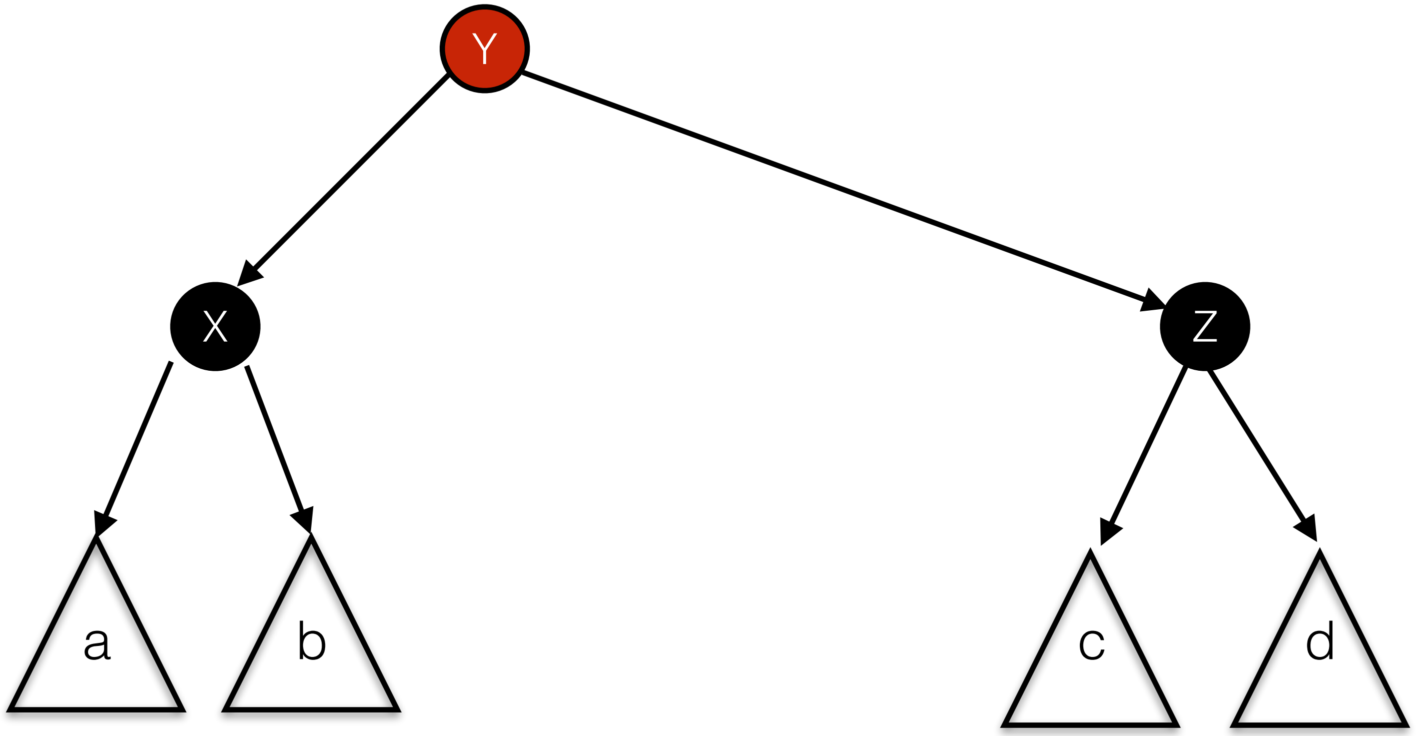

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      …
    }
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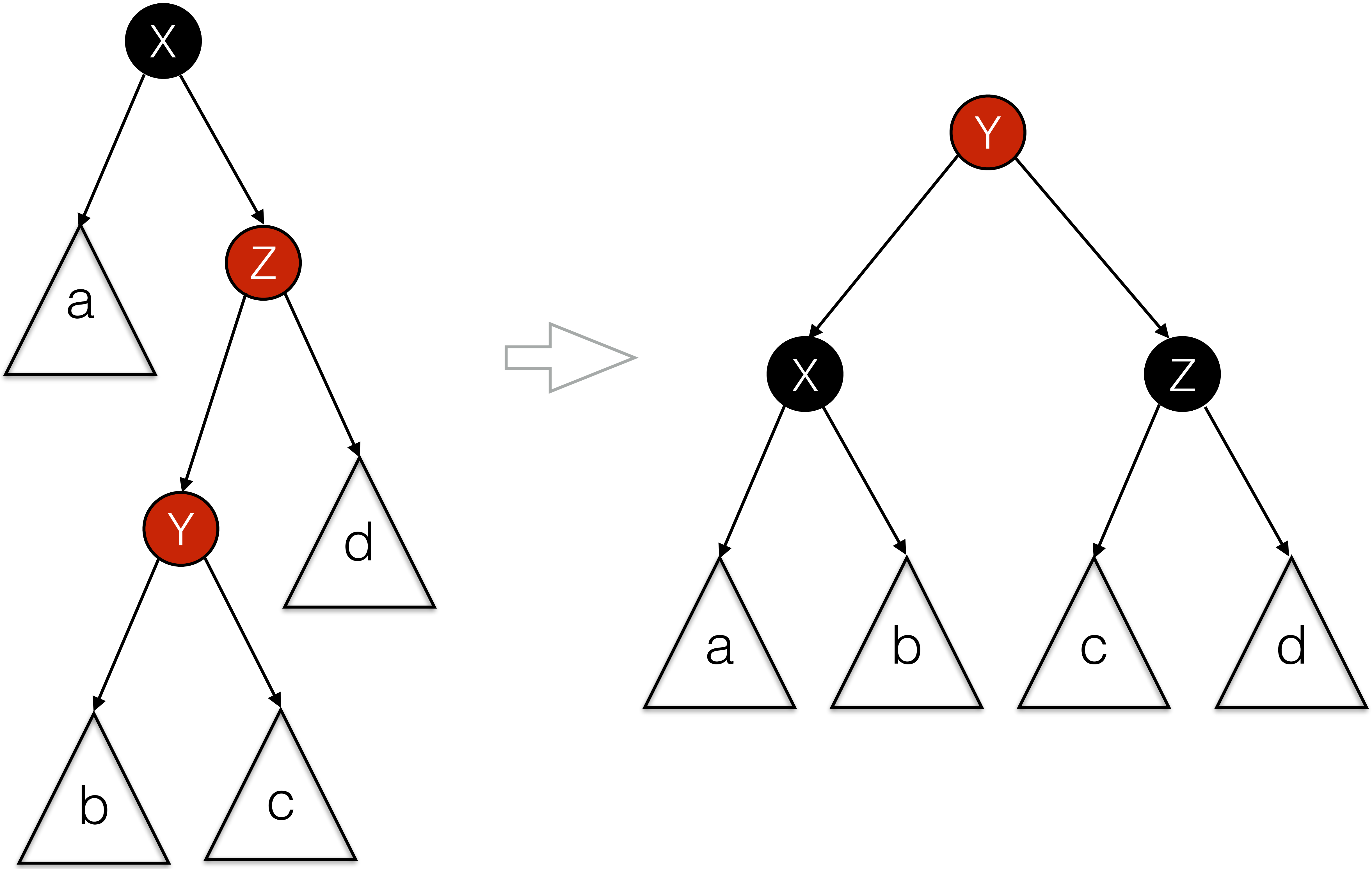
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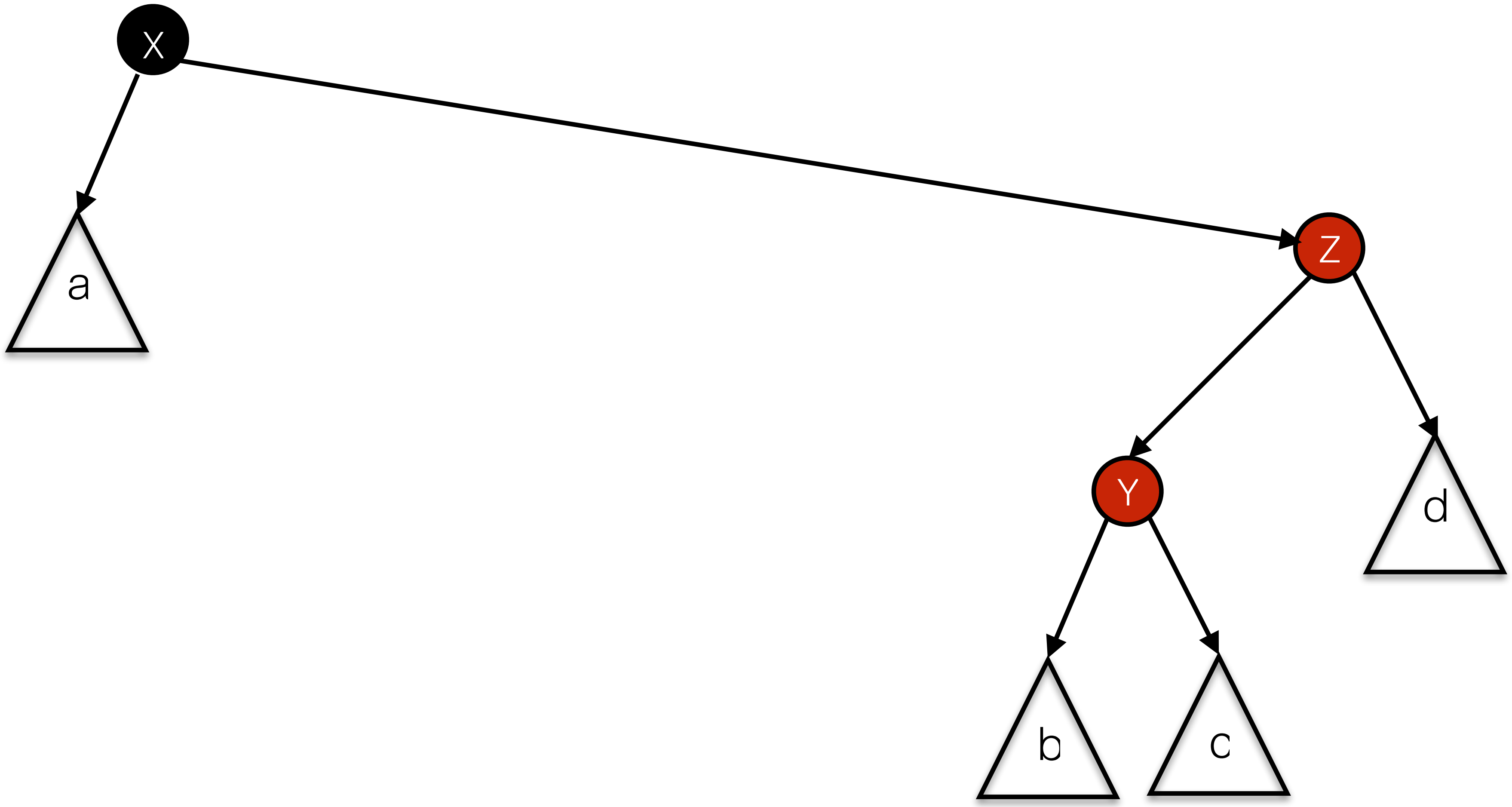

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        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>


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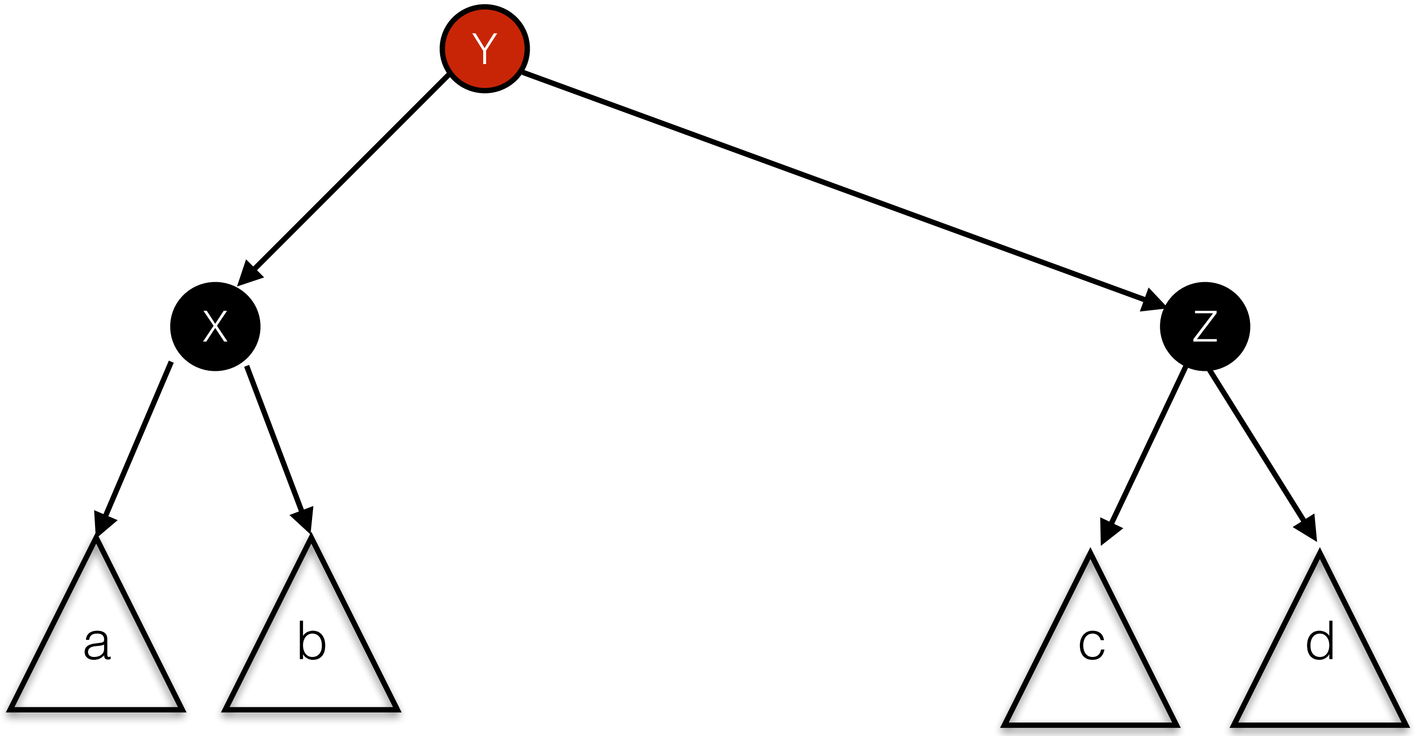

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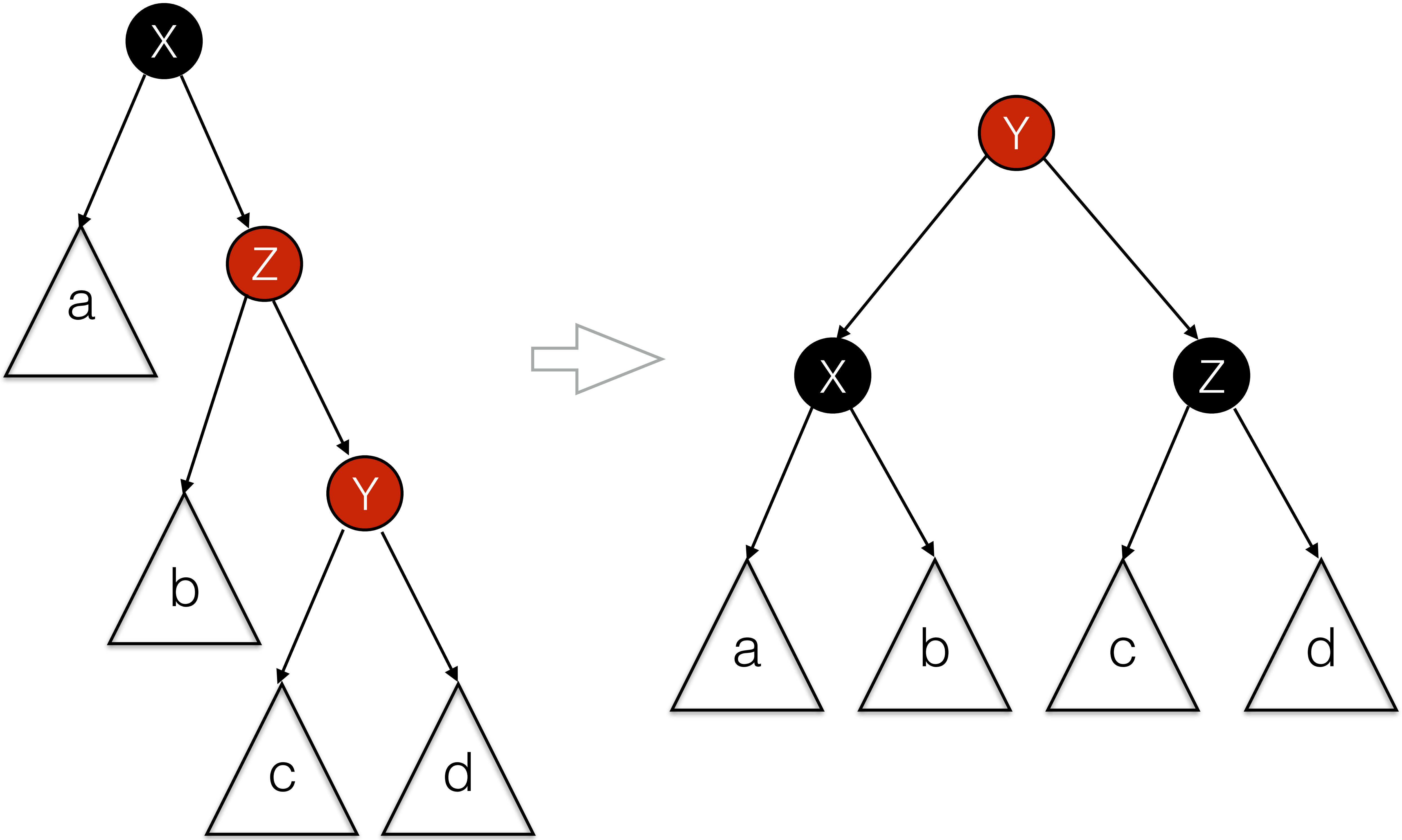
      …
    }
```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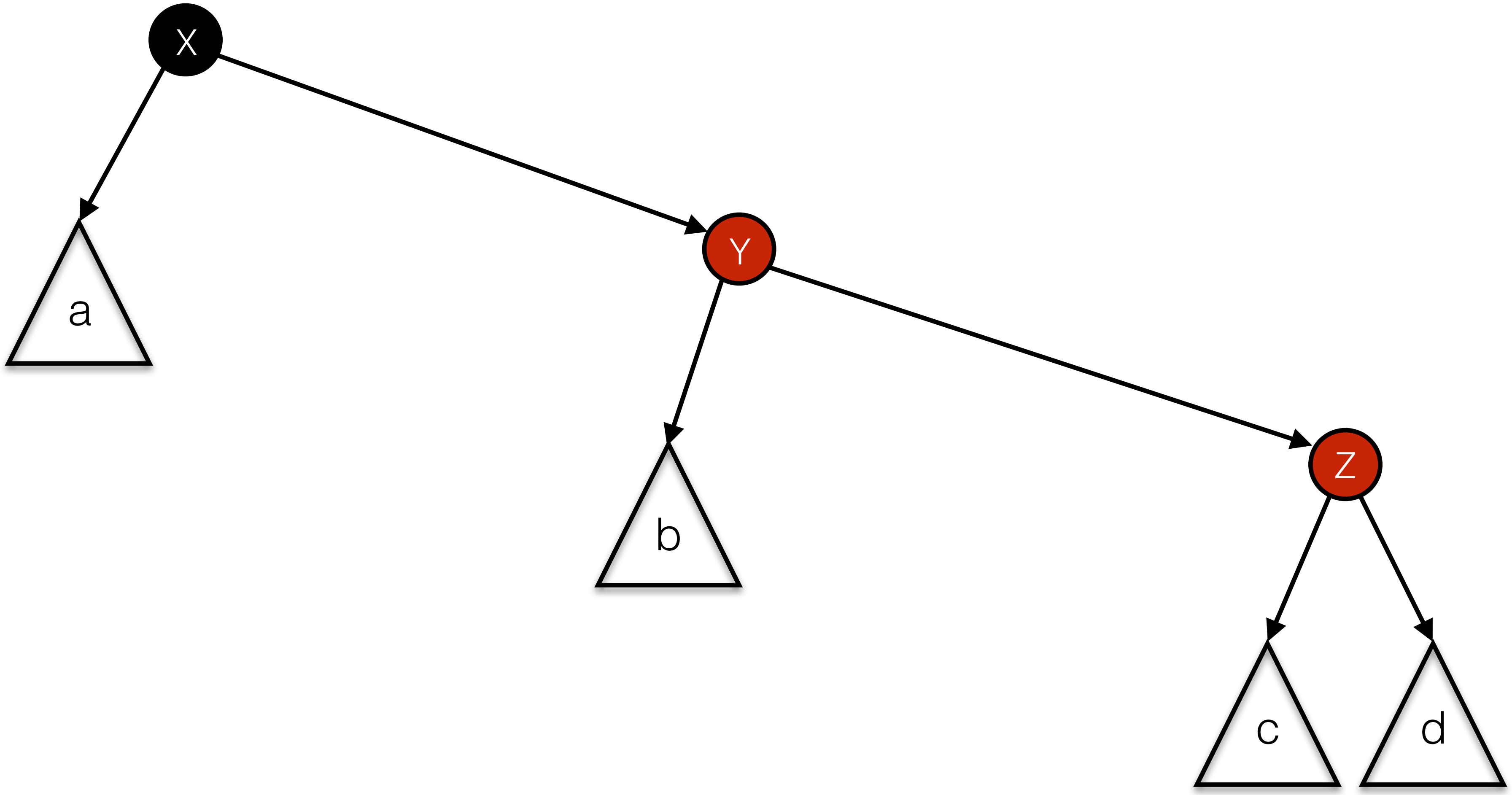

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        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>

      …
```

```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
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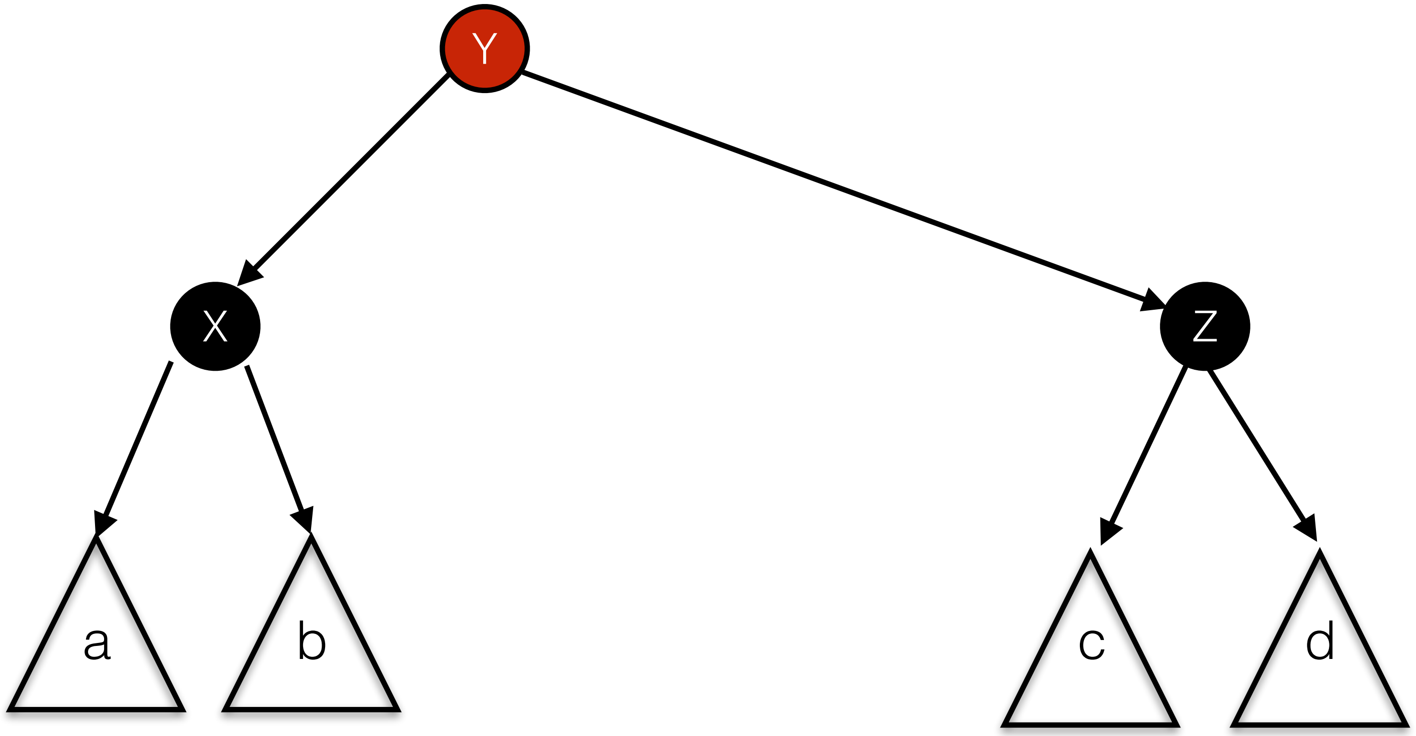
```



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        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      …
```

```
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
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      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      …
    }
  }
```

```scala
def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
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      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case _ => Branch(c, l, x, r)
    }
}
```

Red-Black Trees

```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {
  …
  def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
      case (Black, Branch(Red, Branch(Red, a, x, b), y, c), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>
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      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
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        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case _ => Branch(c, l, x, r)
    }
  }
  …
}
```

Unfortunately, all four consequences are syntactically identical

Red-Black Trees

```
case class Branch[T <: Ordered[T]]
(color: Color, left: Tree[T], element: T, right: Tree[T])
extends Tree[T] {
  …
  def balance(c: Color, l: Tree[T], x: T, r: Tree[T]) = {
    (c, l, x, r) match {
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      case (Black, Branch(Red, a, x, Branch(Red, b, y, c)), z, d) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, Branch(Red, b, y, c), z, d)) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case (Black, a, x, Branch(Red, b, y, Branch(Red, c, z, d))) =>
        Branch(Red, Branch(Black, a, x, b), y, Branch(Black, c, z, d))
      case _ => Branch(c, l, x, r)
    }
  }
  …
}
```

In some languages (such as ML) we could factor this out with "or" patterns

Discussion

- This implementation of red-black trees is dramatically simpler than most imperative approaches:

  - Imperative approaches typically include eight cases, branching on the color of the red parent’s sibling

  - These cases help to avoid some assignment and copying in an imperative setting

Streams

Streams

- Streams are a form of “lazy” sequence

- Inspired by signal-processing systems (such as digital circuits):

  - Components accept *streams* of signals as input, transform their input, and produce streams of signals as outputs

Streams

```
abstract class Stream[+T] {
  def head(): T
  def tail(): Stream[T]
  def map[S](f: T => S): Stream[S]
  def flatMap[S](f: T => Stream[S]): Stream[S]
  def ++[S >: T](that: Stream[S]): Stream[S]
  def withFilter(f: T => Boolean): Stream[T]
  def nth(n: Int): T
}
```

Streams

```
case object NilStream extends Stream[Nothing] {
  def head() = throw new Error()
  def tail() = throw new Error()
  def map[S](f: Nothing => S): Stream[S] = NilStream
  def flatMap[S](f: Nothing => Stream[S]): Stream[S] =
    NilStream
  def ++[S >: Nothing](that: Stream[S]) = that
  def withFilter(f: Nothing => Boolean) = NilStream
  def nth(n: Int) = throw new Error()
}
```

Streams

```
case class ConsStream[+T](head: T, _tail: () => Stream[T])
extends Stream[T] {
  def tail = _tail()
  def map[S](f: T => S): Stream[S] =
    ConsStream(f(head), () => (tail map f))
  def flatMap[S](f: T => Stream[S]): Stream[S] =
    f(current) ++ tail.flatMap(f)
  def ++[S >: T](that: Stream[S]): Stream[S] =
    ConsStream(head, () => tail ++ that)
  …
}
```

Streams

```
case class ConsStream[+T](head: T, _tail: () => Stream[T])
extends Stream[T] {
  …
  def withFilter(f: T => Boolean) = {
    if (f(head)) ConsStream(head, () => tail.withFilter(f))
    else tail.withFilter(f)
  }
  def nth(n: Int) = {
    require (n >= 0)
    if (n == 0) head
    else tail.nth(n - 1)
  }
}
```

Streams

```
def range(low: Int, high: Int): Stream[Int] =
  if (low > high) NilStream
  else ConsStream(low, () => range(low + 1, high))
```

Streams

```
def intsFrom(n: Int): Stream[Int] =
  ConsStream(n, () => intsFrom(n + 1))
```

Streams

```
val nats = intsFrom(0)
```

Streams

```
def fibGen(a: Int, b: Int): Stream[Int] =
  ConsStream(a, () => fibGen(b, a + b))
```

Streams

```
val fibs = fibGen(0, 1)
```

Streams

```
def push(x: Int, ys: Stream[Int]) = {
  ConsStream(x, () => ys)
}
```

Streams

```
def isDivisible(m: Int, n: Int) = (m % n == 0)
```

Streams

```
def isDivisible(m: Int, n: Int) = (m % n == 0)

val noSevens = nats withFilter (isDivisible(_, 7))
```

A Prime Sieve

```
def sieve(stream: Stream[Int]): Stream[Int] =
  ConsStream(stream.head,
             () => sieve(stream.tail withFilter
                           (x => !(isDivisible
                                     (x, stream.head)))))
```

A Stream of Primes

```
val primes = sieve(intsFrom(2))
```

A Stream of Primes

```
> primes.head
res5: Int = 2
> primes.nth(1)
res6: Int = 3
> primes.nth(2)
res7: Int = 5
> primes.nth(3)
res8: Int = 7
```

Streams

```
def add(xs: Stream[Int], ys: Stream[Int]): Stream[Int]
= {
  (xs, ys) match {
    case (NilStream, _) => ys
    case (_, NilStream) => xs
    case (ConsStream(x,f), ConsStream(y,g)) =>
      ConsStream(x + y, () => add(f(), g()))
  }
}
```

Streams

```
def ones(): Stream[Int] = ConsStream(1, ones)
```

Alternative Definition of the Stream of Natural Numbers

```
def nats(): Stream[Int] =
  ConsStream(0, () => add(ones, nats))
```

Alternative Definition of the Fibonacci Stream

```
def fibs(): Stream[Int] =
  ConsStream(0,
            () => ConsStream(1,
                             () => add(fibs.tail, fibs)))
```

Powers of Two

```scala
def scaleStream(c: Int, stream: Stream[Int]): Stream[Int] =
  stream map (_ * c)

def powersOfTwo(): Stream[Int] =
  ConsStream(1, () => scaleStream(2, powersOfTwo))
```

Alternative Definition of the Stream of Primes

```
def primes() =
  ConsStream(2, () => intsFrom(3) withFilter isPrime)

def isPrime(n: Int): Boolean = {
  def iter(next: Stream[Int]): Boolean = {
    if (square(next.head) > n) true
    else if (isDivisible(n, next.head)) false
    else iter(next.tail)
  }
  iter(primes)
}
```

Numeric Integration with Streams

$$S_i = c + \sum_{j=1}^{i} x_j dt$$

Numeric Integration with Streams

```
def integral(integrand: Stream[Double], init: Double,  dt: Double)
= {
  def inner(): Stream[Double] = {
    ConsStream(init,
               () => addStreams(scaleStream(dt,
                                            integrand),
                                inner))
  }
  inner
}
```

Streams and Local State

```
def withdraw(balance: Int, amounts: Stream[Int]):
Stream[Int] = {
  ConsStream(balance,
             () => withdraw(balance - amounts.head,
                            amounts.tail))
}
```

Discussion

- Our modeling of a bank account is a purely functional program without state
- Nevertheless:
  - If a user provides the stream of withdrawals, and
  - The stream of balances is displayed as outputs,
- The system will behave from a user's perspective as a stateful system

Discussion

- The key to understanding this paradox is that the “state” is in the world:
  - The user/bank system is stateful and provides the input stream
  - If we could “step outside” our own perspective in time, we could view our withdrawal stream as another stateless stream of transactions