Comp 311
Functional Programming

Nick Vrvilo, Two Sigma Investments
Robert "Corky" Cartwright, Rice University

September 19, 2017

Design Templates for Abstract Datatypes
(Part 2)

Case Two
We Expect Many New Functions
But Few New Variants

Case 2: We Expect Many New Functions But Few New Variants

- This is a case that traditional functional programming handles well

- Classic example domains: Compilers, theorem provers, numeric algorithms, machine learning

- Declare a top-level function with cases for each data variant

a.k.a., The Visitor Pattern

Again We Turn to Pattern Matching

```scala
val pi = 3.14

def area(shape: Shape) = {
  shape match {
    case Circle(r) => pi * r * r
    case Square(x) => x * x
    case Rectangle(x,y) => x * y
  }
}
```

We Can Define Arbitrary Functions Without Modifying Data Definitions

```scala
def makeLikeFirst(shape0: Shape, shape1: Shape) = {
  (shape0, shape1) match {
    case (Circle(r), Square(s)) => Circle(s)
    case (Circle(r), Rectangle(l,w)) => Circle((l+w)/2)

    case (Square(s), Circle(r)) => Square(r)
    case (Square(s), Rectangle(l,w)) => Square((l+w)/2)

    case (Rectangle(l,w), Circle(r)) => Rectangle(r,r)
    case (Rectangle(l,w), Square(s)) => Rectangle(s,s)

    case _ => shape1
  }
}
```

But A New Data Variant Requires Us To Modify All Functions Over the Datatype

```scala
val pi = 3.14

def area(shape: Shape) = {
  shape match {
    case Circle(r) => pi * r * r
    case Square(x) => x * x
    case Rectangle(x,y) => x * y
    case Triangle(b,h) => b*h/2
  }
}
```

But A New Data Variant Requires Us To Modify All Functions Over the Datatype

```scala
def makeLikeFirst(shape0: Shape, shape1: Shape) = {
  (shape0, shape1) match {
    case (Circle(r), Square(s)) => Circle(s)
    case (Circle(r), Rectangle(l,w)) => Circle((l+w)/2)
    case (Circle(r), Triangle(b,h)) => Circle(b)

    case (Square(s), Circle(r)) => Square(r)
    case (Square(s), Rectangle(l,w)) => Square((l+w)/2)
    case (Square(s), Triangle(b,h)) => Square(b+h/2)

    case (Rectangle(l,w), Circle(r)) => Rectangle(r,r)
    case (Rectangle(l,w), Square(s)) => Rectangle(s,s)
    case (Rectangle(l,w), Triangle(b,h)) => Rectangle(b,h)

    // plus all the cases for Triangle on the left (omitted)
    case _ => shape1
  }
}
```

Sealed Data Types

- Adding the **sealed** keyword to an abstract type indicates that all subclasses of that type are declared in the current compilation unit.

- Provides extra information to the compiler for optimizations and diagnostics

```scala
sealed abstract class Shape
case class Square(length: Double) extends Shape
case class Circle(radius: Double) extends Shape
case class Triangle(base: Double, height: Double)
    extends Shape
```

Sealed Data Types

```scala
object Math {
  val pi = 3.141592653589793
}

sealed abstract class Shape {
  def area: Double = this match {
    // case Square(x) => x * x
    case Circle(r) => Math.pi * r * r
    case Triangle(b, h) => 0.5 * b * h
  }
}
```

warning: match may not be exhaustive.
It would fail on the following input: Square(_)
        def area: Double = this match {

Recursively Defined Datatypes

Recursively Defined Datatypes

- Case classes allow us to combine multiple pieces of a data into a single object

- But sometimes we don't know how many things we wish to combine

- We can use recursion to define datatypes of unbounded size

- This case corresponds to the Composite Design Pattern

Backus-Naur Form
For Lists of Ints

```
List ::= Empty
       | Cons(Int,List)
```

Examples of Lists

```
Empty
Cons(3, Empty)
Cons(3, Cons(1, Empty))
Cons(3, Cons(1, Cons(4, Empty)))
```

Defining Lists With Scala Case Classes

```scala
abstract class List
case object Empty extends List
case class Cons(head: Int, tail: List) extends List
```

Where Do We Put Functions Over Lists?

- We do not expect to define new subtypes of lists

- We do expect to define many new functions over lists

- Similar to our Case Two Design Template for Abstract Datatypes

- Thus, we will start with our pattern matching template

An Example Function for Lists

```scala
def containsZero(xs: List): Boolean = {
  xs match {
    case Empty => false
    case Cons(n, ys) => {
      if (n == 0) true
      else containsZero(ys)
    }
  }
}
```

An Example Function for Lists

```scala
def containsZero(xs: List): Boolean = {
  xs match {
    case Empty => false
    case Cons(n, ys) => (n == 0) || containsZero(ys)
  }
}
```

Generalizing to Our First Template Function for Lists

```scala
def ourFunction(xs: List): Boolean = {
  xs match {
    case Empty => …
    case Cons(n, ys) => … n … ourFunction(ys) …
  }
}
```

Generalizing to Our First Template Function for Lists

```scala
def ourFunction(xs: List): Boolean = {
  xs match {
    case Empty => …
    case Cons(n, ys) => … n … ourFunction(ys) …
  }
}
```

We need to determine our *base case*

Generalizing to Our First Template Function for Lists

```scala
def ourFunction(xs: List): Boolean = {
  xs match {
    case Empty => …
    case Cons(n, ys) => … n … ourFunction(ys) …
  }
}
```

We must determine how to combine these values

Generalizing to Our First Template Function for Lists

```scala
def ourFunction(xs: List): Boolean = {
  xs match {
    case Empty => …
    case Cons(n, ys) => … n … ourFunction(ys) …
  }
}
```

This template is an example of *natural recursion* or *structural recursion*: We recursively decompose and then recombine a computation according to the natural structure of the data.

Filling in the Template

```scala
def containsZero(xs: List): Boolean = {
  xs match {
    case Empty => false
    case Cons(n, ys) => (n == 0) || containsZero(ys)
  }
}
```

Here the base case is easy:
An empty list does not contain zero
(or anything else)

Filling in the Template

```scala
def containsZero(xs: List): Boolean = {
  xs match {
    case Empty => false
    case Cons(n, ys) => (n == 0) || containsZero(ys)
  }
}
```

We break into cases based on the pieces from match: Either our first element *n* is zero or the answer lies with the rest of the list

Another Example: How Many Elements?

```scala
def length(xs: List): Int = {
  xs match {
    case Empty => 0
    case Cons(n, ys) => 1 + length(ys)
  }
}
```

Another Example:
The Sum of the Elements

```scala
def sum(xs: List): Int = {
  xs match {
    case Empty => 0
    case Cons(n, ys) => n + sum(ys)
  }
}
```

Another Example:
The Product of the Elements

```scala
def product(xs: List): Int = {
  xs match {
    case Empty => 1
    case Cons(n, ys) => n * product(ys)
  }
}
```
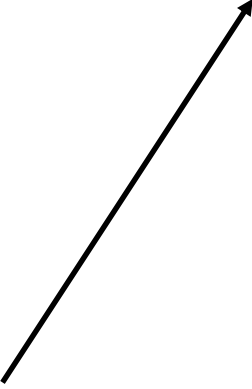
Converting Hours to Seconds

Problem Statement: Given a list of times measured in hours, we want to construct a list of corresponding times measured in seconds

Converting Hours to Seconds

```scala
def hoursToSeconds(xs: List): List = {
  xs match {
    case Empty => Empty
    case Cons(n, ys) => Cons(seconds(n), hoursToSeconds(ys))
  }
}

def seconds(hours: Int) = 3600 * hours
```

Generalizing to a Template

```
def ourFunction(xs: List): List = {
  xs match {
    case Empty => …
    case Cons(n, ys) => Cons(…n…,
                              ourFunction(ys))
  }
}
```

Really, this is the same template as
before, but now Cons is our combining
operation

The Natural Numbers

```
Nat ::= 0
      | Next(Nat)
```

The Natural Numbers

```
Nat ::= 0
        | Next(Nat)
```

Here we are between Cases One and Two for Abstract Datatypes:

- No new variants expected
- Many new functions expected
- But some basic functions are intrinsic to the type

Defining The Natural Numbers in Scala

```scala
abstract class Nat
case object Zero extends Nat
case class Next(n: Nat) extends Nat
```

Defining The Natural Numbers in Scala

```scala
abstract class Nat {
  def +(n: Nat): Nat
  def *(n: Nat): Nat
}
```
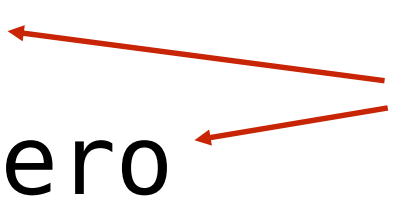
Defining The Natural Numbers in Scala

```scala
case object Zero extends Nat {
  def +(n: Nat) = n
  def *(n: Nat) = Zero
}


case class Next(n: Nat) extends Nat {
  def +(m: Nat) = Next(n + m)
  def *(m: Nat) = m + (n * m)
}
```
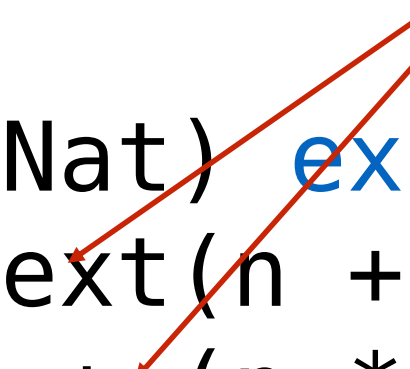
Defining The Natural Numbers in Scala

```scala
case object Zero extends Nat {
    def +(n: Nat) = n
    def *(n: Nat) = Zero
}

case class Next(n: Nat) extends Nat {
    def +(m: Nat) = Next(n + m)
    def *(m: Nat) = m + (n * m)
}
```

Again we have natural recursion: base case, recursion, combination

Example Reduction
(3 + 2)

```
Next(Next(Next(Zero)) + Next(Next(Zero)) ↦
Next(Next(Next(Zero)) + Next(Next(Zero))) ↦
Next(Next(Next(Zero) + Next(Next(Zero)))) ↦
Next(Next(Next(Zero + Next(Next(Zero))))) ↦
    Next(Next(Next(Next(Next(Zero)))))
```

Factorial

```
def factorial(n: Nat): Nat = {
  n match {
    case Zero => Next(Zero)
    case Next(m) => n * factorial(m)
  }
}
```

Transferring The Pattern To Ints

```scala
def factorial(n: Int): Int = {
  require (n >= 0)

  if (n == 0) 1
  else n * factorial(n - 1)

} ensuring (_ > 0)
```
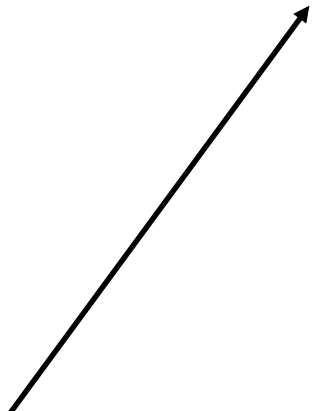
Combining Via Auxiliary Functions

Combining Via Auxiliary Functions

- As our examples with natural numbers shows, it is often desirable to define the *combining operation* of a natural recursion as an auxiliary function

- We can apply this insight to lists and use our template to cover yet more cases

Sorting Lists

```scala
def sort(xs: List): List = {
  xs match {
    case Empty => Empty
    case Cons(n, ys) => insert(n, sort(ys))
  }
}
```

We need to explain how to
insert into a sorted list

Insertion

```scala
def insert(n: Int, xs: List): List = {
  xs match {
    case Empty => Cons(n, Empty)
    case Cons(m, ys) => {
      if (n <= m) Cons(n, xs)
      else Cons(m, insert(n, ys))
    }
  }
}
```

Insertion

```scala
def insert(n: Int, xs: List): List = {
  xs match {
    case Empty => Cons(n, Empty)
    case Cons(m, ys) => {
      if (n <= m) Cons(n, xs)
      else Cons(m, insert(n, ys))
    }
  }
}
```

This parameter is not traversed,
but is used for combination and comparison
Other functions follow this pattern.

Appending Two Lists

```scala
abstract class List {
  /**
    * Returns a new list with the elements of
    * this list appended to the given list.
    */
  def ++(ys: List): List
}
```

Appending Two Lists

```scala
case object Empty extends List {
  def ++(ys: List) = ys
}
```

Appending Two Lists

```scala
case class Cons(first: Int, rest: List) extends List {
  def ++(ys: List) = Cons(first, rest ++ ys)
}
```