Comp 311
Functional Programming

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Announcements

- Homework 1 was due today before class
  - Submission date determined from SVN timestamps
    - Each student has 7 slip days for the whole semester
    - Can use up to 3 slip days per assignment
- Homework 2 is out today (PDF on Piazza)
  - Due in 2 weeks

Additional Syntax for Homework 2

Exception Handling

- Scala handles error by *throwing* exceptions
- An exception can be *caught* by a scope that is a *dynamic* parent of the throwing scope
- We'll delay the detailed reduction semantics for try/catch until a later date

```
try  // expression...
catch {
  case Pattern => // expression...
  // zero or more additional cases...
}
```

Try/Catch Examples

Patterns can be restricted to only match a specific type

```
try {
  require(false)
}
catch {
  case e: IllegalArgumentException =>
    assertEquals(e.getMessage,
                 "requirement failed")
}
```

Try/Catch Examples

```
try abs(Int.MinValue)
catch {
  // precondition failure
  case e: IllegalArgumentException =>
    assertEquals(e.getMessage,
                 "requirement failed")
  // postcondition failure
  case e: AssertionError =>
    assertEquals(e.getMessage,
                 "assertion failed")
}
```

Custom Error Messages

```
try {
  require(false, "my message")
}
catch {
  case e: IllegalArgumentException =>
    assertEquals(e.getMessage,
        "requirement failed: my message")
}
```

Custom Error Messages

```
def abs(x: Int) = {
  if (x < 0) -x else x
} ensuring(_>0, "negative absolute value")
```

Non-trivial Postconditions

```
def add(x: Int, y: Int) = {
  x + y
} ensuring(result => {
    if (x >= 0 && y >= 0)
      result >= max(x, y)
    else if (x < 0 && y < 0)
      result < min(x, y)
    else
      result >= min(x, y)
}, "integer overflow during addition")
```

String Formatting

Scala strings support `printf`-style formatting:

```
"Customer %s ordered %d units"
    .format(accountName, 500)
```

String Interpolation

Scala strings also support *interpolation* of values:

Prefix the string with an `s`

```
s"My name is $name."

s"My name is ${name}.“

s"Max value: ${max(x, y)}."
```

Prefix variable names with $

General expressions must be wrapped in braces

Parametric Polymorphism (Parametric/Generic Types)

Parametric Types

- We have defined two forms of lists: lists of ints and lists of shapes
- Many computations useful for one are useful for the other:
  - Map, reduce, filter, etc.
- It would be better to define lists and their operations once for all of these cases

Parametric Types

- Higher-order functions take functions as arguments and return functions as results

- Likewise, *parametric types*, a.k.a., a *generic types*, takes types as arguments and return types as results

Parametric Lists

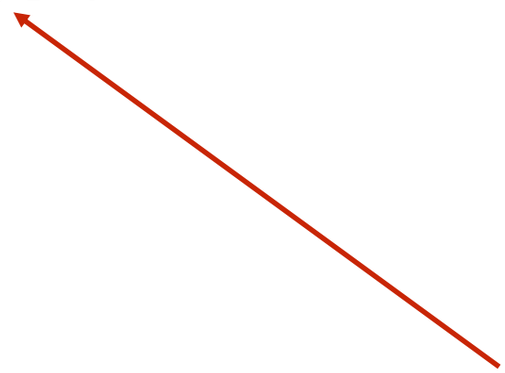
- Every application of this parametric type to an argument yields a new type:

```
abstract class List[T] {
  def ++(ys: List[T]): List[T]
}
```

Parametric Lists

- Every application of this parametric type to an argument yields a new type:

```
abstract class List[T <: Any] {
  def ++(ys: List[T]): List[T]
}
```



- We augment the declarations of type parameters to permit an upper bound on all instantiations of a parameter
  - By default, the bound is Any

Syntax of Parametric Class Definitions

```
<modifiers> class C[T1 <: N,..,TN <: N] extends N {
  <ordinary class body>
}
```

- We denote type parameters as T1, T2, etc.
- We denote all other types with N, M, etc.

Syntax of Parametric Class Definitions

```
<modifiers> class C[T1 <: N,..,TN <: N] extends N {
  <ordinary class body>
}
```

- Declared type parameters T1, ..., TN are in scope throughout the entire class definition, including:
  - The bounds of type parameters
  - The extends clause
- Object definitions must not be parametric

Parametric Lists

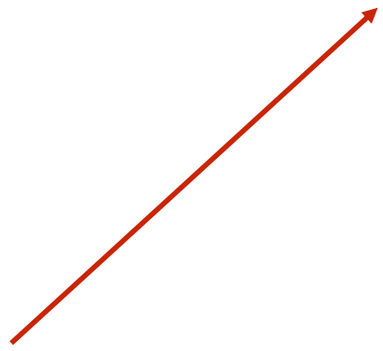
- Every application of this parametric type yields a new type:

List[Int]
List[String]
List[List[Double]]
etc.

Parametric Lists

- Every application (a.k.a., *instantiation*) of this parametric type yields a new type:

```
abstract class List[T] {
  def ++(ys: List[T]): List[T]
}
```



Note that our parametric type can be instantiated with type parameters, including its own!

Parametric Lists

```
case class Empty[S]() extends List[S] {
  def ++(ys: List[S]) = ys
}

case class Cons[T](head: T, tail: List[T]) extends List[T] {
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)
}
```

Parametric Lists

```
case class Empty[S]() extends List[S] {
  def ++(ys: List[S]) = ys
}

case class Cons[T](head: T, tail: List[T]) extends List[T] {
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)
}
```

Our definition requires a separate type Empty[S] for every instantiation of S. Thus we must define Empty as a class rather than an object.

Type Environments

- To explain how to type check expressions in the context of parametric types, we must introduce the notion of *environments*

- We define a type parameter environment to hold a collection of zero or more type parameter declarations with their bounds

- Type environments can be extended with more declarations

Type Checking a Class Definition

- To type check a parametric class definition:
  - Check the declarations of the class in a new type parameter environment that extends the enclosing environment with all its type parameters

Type Checking a Function Definition

- To type check a function definition in environment E:
  - Check that the types of all parameters are *well-formed*
  - Find the type of the body of the function, substituting occurrences of parameters with their types
  - Ensure that the type of the body is a subtype of the declared return type (in environment E)

Well-Formedness of Types

- A type is well-formed in environment E iff:
  - If it is a well-defined non-parametric type
  - It is a type parameter T in environment E
  - It is an instantiation of a defined parametric type and:
    - All of its type arguments are well-formed types in E
    - All of its type arguments respect the bounds on their corresponding type parameters

Subtyping With Environments

- It is non-sensical to compare types in separate type environments:

```
case class Empty[S]() extends List[S] {
  def ++(ys: List[S]) = ys
}

case class Cons[T](head: T, tail: List[T]) extends List[T] {
  def ++(ys: List[T]) = Cons[T](head, tail ++ ys)
}
```

  - Is S a subtype of T?

Subtyping With Environments

- We must modify our subtyping rules to refer to an environment E:

- S <: S in E

- If S <: T in E and T <: U in E then S <: U in E

Subtyping With Environments

- If:
- class C[T1,..,TN] extends D[U1,...UM]
- and X1,...,XN are well-formed in E
- then C[X1,...XN] <: D[U1,...,UM][T1↦X1,...,TN↦XN] in E

Subtyping With Environments

- If:
  - class C[T1,..,TN] extends D[U1,…UM]
  - and X1,…,XN are well-formed in E
  - then C[X1,…XN] <: D[U1,…,UM][T1↦X1,…,TN↦XN] in E

We use this notation to indicate safe substitution of T1 for X1, … TN for XN in D[U1,…,UM]

Covariance

- Can one instantiation of a parametric type be a subtype of another?

- Currently our rules allow this only in the reflexive case:

List[Int] <: List[Int] in E

Covariance

- It would be useful to allow some instantiations to be subtypes of another
- For example, we would like it to be the case that:

List[Int] <: List[Any]

Covariance

- In general, we say that a parametric type C is covariant with respect to its type parameter S if:

S <: T in E

implies

C[S] <: C[T] in E

- We must be careful that such relationships do not break the soundness of our type system

Covariance

- For a parametric type such as:

```
abstract class List[T <: Any] {
  def ++(ys: List[T]): List[T]
}
```

- And types S and T, such that S <: T in some environment E:
  - What must we check about the body of class List to allow for List[S] <: List[T] in E?

Covariance

- Consider instantiations for types String and Any:

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] {
  def ++(ys: List[String]): List[String]
}
```

Covariance

- If these were ordinary classes connected by an extends class:

- We would need to ensure that the overriding definition of ++ in class List[String] was compatible with the overridden definition in List[Any]

Covariance

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] extends List[Any] {
  def ++(ys: List[String]): List[String]
}
```

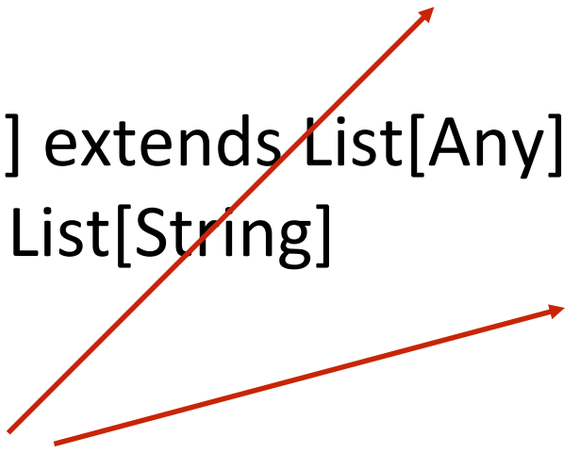
Covariance

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] extends List[Any] {
  def ++(ys: List[String]): List[String]
}
```

But if List[String] <: List[Any] in E
then this is not a valid override

Covariance

```
abstract class List[Any] {
  def ++(ys: List[Any]): List[Any]
}
abstract class List[String] extends List[Any] {
  def ++(ys: List[String]): List[String]
}
```



On the other hand, the return types are not problematic

Covariance

- From our example, we can glean the following rule:
- We allow a parametric class C to be covariant with respect to a type parameter T so long as T does not appear in the types of the method parameters of C

Covariance

```
abstract class List[+T] {}
```

- We stipulate that a parametric type is covariant in a parameter T by prefixing a + at the definition of T

- (We will return to our definition of append later)

Covariance

```
case object Empty extends List[Nothing] {
}

case class Cons[+T](head: T, tail: List[T])
extends List[T] {
}
```

Covariance

```
case object Empty extends List[Nothing] {
}

case class Cons[+T](head: T, tail: List[T])
extends List[T] {
}
```

low we can define Empty as an object that extends the bottom of the List type

Covariance and Append

- The problem with our original declaration of append was that it was not general enough:
  - There is no reason to require that we always append lists of identical type
  - Really, we can append a List[S] for any supertype of our List[T]
  - The result will be of type List[S]

Lower Bounds on Type Parameters

- Thus far, we have allowed type parameters to include upper bounds:

```
T <: S
```

- They can also include lower bounds:

```
T >: U
```

- Or they can include both:

```
T >: S <: U
```

Parametric Functions

- Just as we can add type parameters to a class definition, we can also add them to a function definition

- The type parameters are in scope in the header and body of the function

Covariance and Append

```
abstract class List[+T] {
  def ++[S >: T](ys: List[S]): List[S]
}
```

```
case object Empty extends List[Nothing] {
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}
```

```
case class Cons[+T](head: T, tail: List[T])
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```

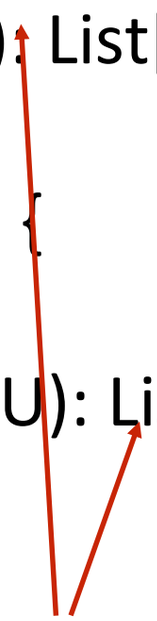
Map Revisited

```
abstract class List[+T] {
  …
  def map[U](f: T => U): List[U]
}
```

Why is this occurrence of T acceptable?

We Consider Specific Instantiations

```
abstract class List[Any] {
  …
  def map[U](f: Any => U): List[U]
}
abstract class List[String] {
  …
  def map[U](f: String => U): List[U]
}
```



Then List[String] is an acceptable subtype of List[Any]
provided that (String => U) >: (Any => U)
which requires that String <: Any.

Generalizing Our Rules

- In our example, type parameter T occurs as the parameter of an arrow type:
- (String => U) >: (Any => U) in E provided:
  - String <: Any in E
  - U <: U in E
- So subtype List[String] <: List[Any] is permitted

To Check Variance, We Annotate Each Type Position With A *Polarity*

- Recursively descend a class definition:
  - At top level, all positions are positive
  - Polarity is flipped at method parameter positions
  - Polarity is flipped at method type parameter positions
  - Polarity is flipped at arrow type parameter positions

Annotating Polarity

```
abstract class List[+T] {
  def ++[S- >: T+](ys: List[S-]): List[S+]
  def map[U-](f: T+ => U-): List[U+]
}
```

We Generalize Our Rules for Checking Variance As Follows

- Covariant type parameters (declared with +) are allowed to occur only in positive locations

- Type parameters with no annotation are allowed to be used in all locations

Contravariance

Contravariance

- In general, we say that a parametric type C is contravariant with respect to its type parameter S if:

S <: T in E

implies

C[T] <: C[S] in E

- We must be careful that such relationships do not break the soundness of our type system

Contravariance

- Syntactically, contravariant type parameter declarations are annotated with a minus sign:

```
case class F[-A,+B]
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- Covariant type parameters (declared with +) are allowed to occur only in positive locations

- Type parameters with no annotation are allowed to be used in all locations

- Contravariant type parameters are allowed to occur only in negative locations

An Example of How We Might Use Contravariant Type Parameters

```
abstract class Function1[-S,+T] {
  def apply(x:S): T
}
```

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```
case object Empty extends List[Nothing] {
  …
  def map[U](f: Nothing => U) = Empty
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Map Revisited

```
case class Cons[+T](head: T, tail: List[T])
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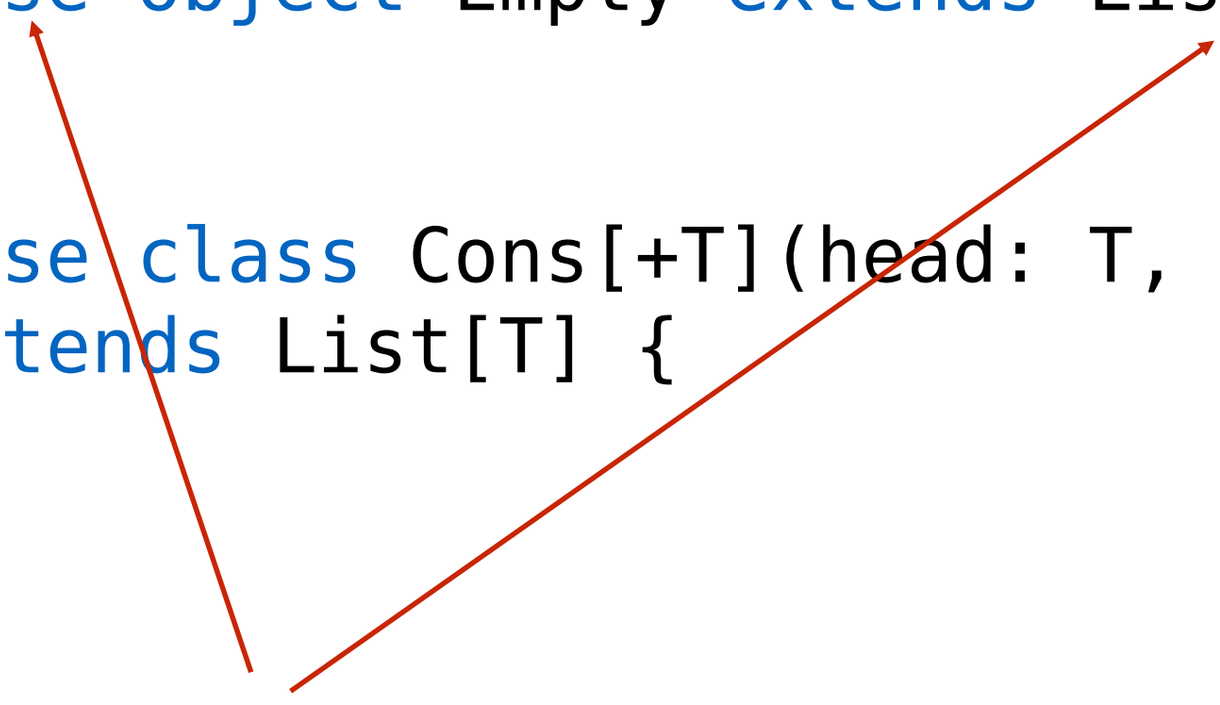

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Annotating Polarity

```
abstract class List[+T] {
  def ++[S⁻ >: T⁺](ys: List[S⁻]): List[S⁺]
  def map[U⁻](f: T⁺ => U⁻): List[U⁺]
}
```

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- Covariant type parameters (declared with +) are allowed to occur only in positive locations

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Contravariance

Contravariance

- In general, we say that a parametric type C is contravariant with respect to its type parameter S if:

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  - Polarity is flipped at arrow type parameter positions
  - Polarity is flipped at positions of contravariant type parameters

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```
abstract class List[+T] {
  def ++[S⁻ >: T⁺](ys: List[S⁻]): List[S⁺]
  def map[U⁻](f: T⁺ => U⁻): List[U⁺]
}
```

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- Contravariant type parameters are allowed to occur only in negative locations

An Example of How We Might Use Contravariant Type Parameters

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abstract class Function1[-S,+T] {
  def apply(x:S): T
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case object Empty extends List[Nothing] {
  …
  def map[U](f: Nothing => U) = Empty
}
```

Map Revisited

```
case class Cons[+T](head: T, tail: List[T])
extends List[T] {
  …
  def map[U](f: T => U) =
    Cons(f(head), tail.map(f))
}
```

Syntactic Sugar: Currying

- Scala provides special syntax for defining a function that immediately returns another function:

```
def f(x0:T1,…,xN:TN) = (y0:U1,…,yM:UM) => expr
```

can be written as:

```
def f(x0:T1,…,xN:TN) (y0:U1,…,yM:UM) = expr
```

- Defining a function in this way is called “currying” after the computer scientist Haskell Curry

Reduce Revisited

```
abstract class List[+T] {
  …
  def foldLeft[S >: T](x: S)(f: (S, S) => S): S
  def foldRight[S >: T](x: S)(f: (S, S) => S): S
}
```

Reduce Revisited

```
case object Empty extends List[Nothing] {
  …
  def foldLeft[S](x: S)(f: (S, S) => S) = x
  def foldRight[S](x: S)(f: (S, S) => S) = x
}
```

Reduce Revisited

```
case class Cons[+T](head: T, tail: List[T])
extends List[T] {
  …
  def foldLeft[S >: T](x: S)(f: (S, S) => S) =
    tail.foldLeft(f(x, head), f)

  def foldRight[S >: T](x: S)(f: (S, S) => S) =
    f(tail.foldRight(x, f), head)
  }
}
```

Reduce Revisited

```
  def foldLeft[S >: T](x: S)(f: (S, S) => S) =
    tail.foldLeft(f(x, head), f)

Cons(1,Cons(2,Cons(3,Empty))).foldLeft(0)(_+_) ↦
Cons(2,Cons(3,Empty)).foldLeft(0 + 1, _+_) ↦
Cons(2,Cons(3,Empty)).foldLeft(1, _+_) ↦
Cons(3,Empty).foldLeft(1 + 2, _+_) ↦
Cons(3,Empty).foldLeft(3, _+_) ↦
Empty.foldLeft(3 + 3, _+_) ↦
Empty.foldLeft(6, _+_) ↦
6
```

Reduce Revisited

```
def foldRight[S >: T](x: S)(f: (S, S) => S) =
  f(tail.foldRight(x, f), head)
```

```
Cons(1,Cons(2,Cons(3,Empty))).foldRight(0)(_+_) ↦
Cons(2,Cons(3,Empty)).foldRight(0, _+_) + 1 ↦
Cons(3,Empty).foldLeft(0, _+_) + 2 + 1 ↦
Empty.foldLeft(0, _+_) + 3 + 2 + 1 ↦
0 + 3 + 2 + 1 ↦
6
```

Reduce Revisited

```
abstract class List[+T] {
  …
  def reduce[S >: T](f: (S, S) => S): S
}
```

We can elide a zero element for the reduction provided that the list is non-empty

Reduce Revisited

```
case object Empty extends List[Nothing] {
  …
  def reduce[S](f: (S, S) => S) =
    throw ReduceError
}
```

case object ReduceError extends Error

Reduce Revisited

```
case class Cons[+T](head: T, tail: List[T])
extends List[T] {
  …
  def reduce[S >: T](f: (S, S) => S) =
    tail.foldLeft[S](head)(f)
}
```

We explicitly instantiate the type parameter to foldLeft. Without this, type inference will instantiate the type parameter based on the static type of head (which is T) and then signal an error that f is not of type (T, T) => T.

Forall and Exists

```
abstract class List[+T] {
  …
  def forall(p: T => Boolean) =
    map(p).foldLeft(true, _&&_)

  def exists(p: T => Boolean) =
    map(p).foldLeft(false, _||_)
}
```

Length

```
abstract class List[+T] { …
  def length: Int
}
case object Empty extends List[Nothing] { …
  def length = 0
}
case class Cons[+T](head: T, tail: List[T])
extends List[T] { …
  def length = map((_:T) => 1).reduce(_+_)
}
```

In what real contexts could we justify this definition of length?

Pointwise Addition

```
def pointwiseAdd(xs: List[Int], ys: List[Int]): List[Int] = {
  require (xs.length == ys.length)
  (xs, ys) match {
    case (Empty, Empty) => Empty
    case (Cons(x1, xs1), Cons(y1, ys1)) =>
      Cons(x1 + y1, pointwiseAdd(xs1,ys1))
  }
}
```

Generalizing to ZipWith

```
// in class List:
def zipWith[U,V](f: (T, U) => V)(that: List[U]): List[V] = {
  require (this.length == that.length)

  (this, that) match {
    case (Empty, Empty) => Empty
    case (Cons(x1,xs1), Cons(y1,ys1)) =>
      Cons(f(x1,y1), xs1.zipWith(f)(ys1))
  }
}
```

Defining The Zip Function

```
// in class List:
def zip[U](that: List[U]) = zipWith((_, _: U))(that)
```

Defining Flatten

```
def flatten[S](xs: List[List[S]]) = {
  xs.foldLeft(Empty)(_++_)
}
```

Defining Flatten

```
def flatten[S](xs: List[List[S]]) = {
  xs.foldLeft(Empty)(_ ++ _)
}
```

Because of the specific type of List needed,
we define as a top level function

Defining FlatMap

```
abstract class List[+T] {
  …
  def flatMap[S](f: T => List[S]) =
    flatten(this.map(f))
  }
}
```

Defining FlatMap

```
abstract class List[+T] {
  …
  def flatMap[S](f: T => List[S]) =
    flatten(this.map(f))
  }
}
```

In contrast to flatten, our flatMap function can be defined on arbitrary lists

Defining FlatMap

- These definitions suggest that flatMap is the best thought of as the more primitive notion

- We can define flatMap as a method on lists directly and then define flatten in terms of it

Defining FlatMap

```
abstract class List[+T] { …
  def flatMap[S](f: Nothing => List[S]): List[S]
}

case object Empty extends List[Nothing] { …
  def flatMap[S](f: Nothing => List[S]) = Empty
}

case class Cons[+T](head: T, tail: List[T])
extends List[T] { …
  def flatMap[S](f: T => List[S]) =
    f(head) ++ tail.flatMap(f)
}
```

Defining Filter

```
abstract class List[+T] {
  …
  def filter[U](p: T => Boolean): List[T]
}
```

Defining Filter

```
case object Empty extends List[Nothing] {
  …
  def filter[U](p: T => Boolean) = Empty
}
```

Defining Filter

```
case class Cons[+T](head: T, tail: List[T])
extends List[T] {
  …
  def filter[U](p: T => Boolean) = {
    if (p(head)) Cons(head, tail.filter(p))
    else tail.filter(p)
  }
}
```

For Expressions

For Expressions

- As with all expressions, for expressions reduce to a value
- The value reduced to is a collection
- The type of collection produced depends on the types of collections iterated over
- Each iteration produces a value to include in the resulting collection

Many Maps and Filters Can Be Expressed Using For Expressions

```
for (x <- xs) yield square(x) + 1
```

Many Maps and Filters Can Be Expressed Using For Expressions

```
for (x <- xs) yield square(x) + 1
```

We call this a generator

Many Maps and Filters Can Be Expressed Using For Expressions

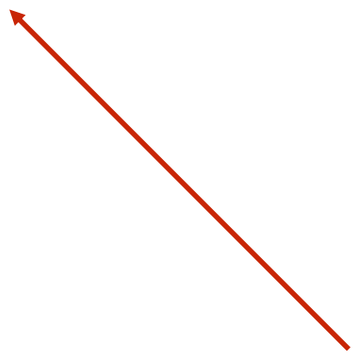
```
for clauses yield body
```

Many Maps and Filters Can Be Expressed Using For Expressions

```
for (i <- 1 to 10) yield square(i) + 1
```

Many Maps and Filters Can Be Expressed Using For Expressions

```
for (i <- 0 until 10) yield square(i) + 1
```



Does not include 10

Many Maps and Filters Can Be Expressed Using For Expressions

```
// BAD FORM
for (i <- 0 until xs.length)
  yield square(xs.nth(i)) + 1
```

Many Maps and Filters Can Be Expressed Using For Expressions

```
// Write this instead
for (x <- xs)
  yield square(x) + 1
```