Comp 311
Functional Programming

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Recursively Defined Datatypes

Recursively Defined Datatypes

- Case classes allow us to combine multiple pieces of a data into a single object

- But sometimes we don't know how many things we wish to combine

- We can use recursion to define datatypes of unbounded size

- This case corresponds to the Composite Design Pattern

Backus-Naur Form
For Lists of Ints

```
List ::= Empty
       | Cons(Int,List)
```

Examples of Lists

```
Empty
Cons(3, Empty)
Cons(3, Cons(1, Empty))
Cons(3, Cons(1, Cons(4, Empty)))
```

Defining Lists With Scala Case Classes

```
sealed abstract class List
case object Empty extends List
case class Cons(head: Int, tail: List) extends List
```

Where Do We Put Functions Over Lists?

- We do not expect to define new subtypes of lists
- We do expect to define many new functions over lists
- Similar to our Case Two Design Template for Abstract Datatypes
- Thus, we will start with our pattern matching template

An Example Function for Lists

```
def containsZero(xs: List): Boolean = {
  xs match {
    case Empty => false
    case Cons(n, ys) => {
      if (n == 0) true
      else containsZero(ys)
    }
  }
}
```

An Example Function for Lists

```
def containsZero(xs: List): Boolean = {
  xs match {
    case Empty => false
    case Cons(n, ys) => (n == 0) || containsZero(ys)
  }
}
```

Generalizing to Our First Template Function for Lists

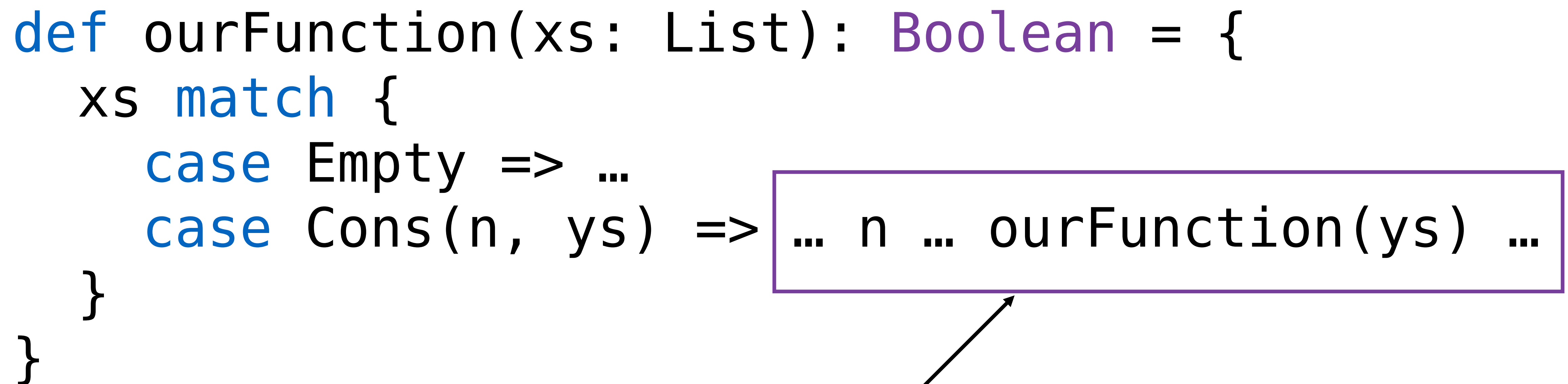
```
def ourFunction(xs: List): Boolean = {
  xs match {
    case Empty => …
    case Cons(n, ys) => … n … ourFunction(ys) …
  }
}
```

Generalizing to Our First Template Function for Lists

```
def ourFunction(xs: List): Boolean = {
  xs match {
    case Empty => …
    case Cons(n, ys) => … n … ourFunction(ys) …
  }
}
```

We need to determine our *base case*

Generalizing to Our First Template Function for Lists



Generalizing to Our First Template Function for Lists

```
def ourFunction(xs: List): Boolean = {
  xs match {
    case Empty => …
    case Cons(n, ys) => … n … ourFunction(ys) …
  }
}
```

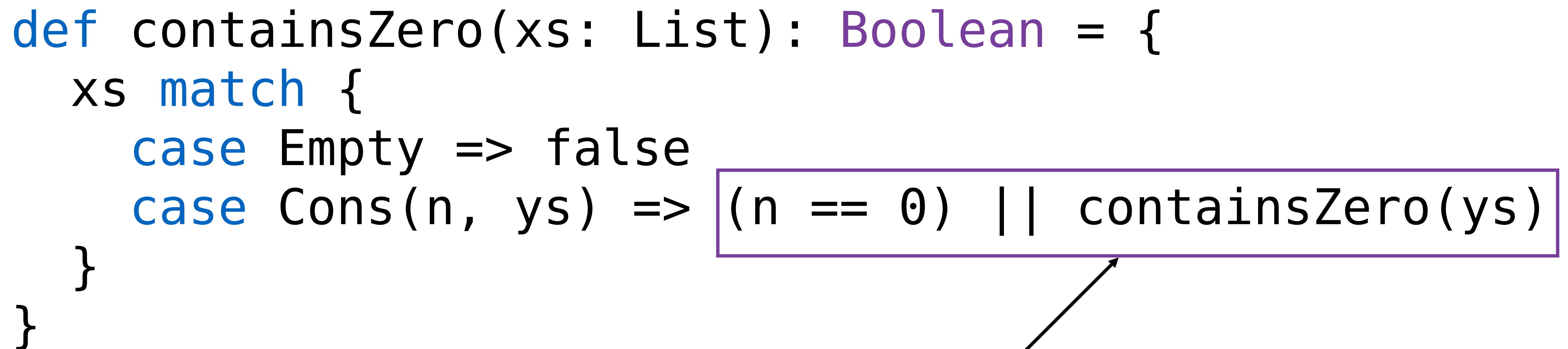
This template is an example of *natural recursion* or *structural recursion*: We recursively decompose and then recombine a computation according to the natural structure of the data.

Filling in the Template

```
def containsZero(xs: List): Boolean = {
  xs match {
    case Empty => false
    case Cons(n, ys) => (n == 0) || containsZero(ys)
  }
}
```

Here the base case is easy:
An empty list does not contain zero
(or anything else)

Filling in the Template



We break into cases based on the pieces from match: Either our first element *n* is zero or the answer lies with the rest of the list

Another Example: How Many Elements?

```
def length(xs: List): Int = {
  xs match {
    case Empty => 0
    case Cons(n, ys) => 1 + length(ys)
  }
}
```

Another Example: The Sum of the Elements

```
def sum(xs: List): Int = {
  xs match {
    case Empty => 0
    case Cons(n, ys) => n + sum(ys)
  }
}
```

Another Example: The Product of the Elements

```
def product(xs: List): Int = {
  xs match {
    case Empty => 1
    case Cons(n, ys) => n * product(ys)
  }
}
```

Converting Hours to Seconds

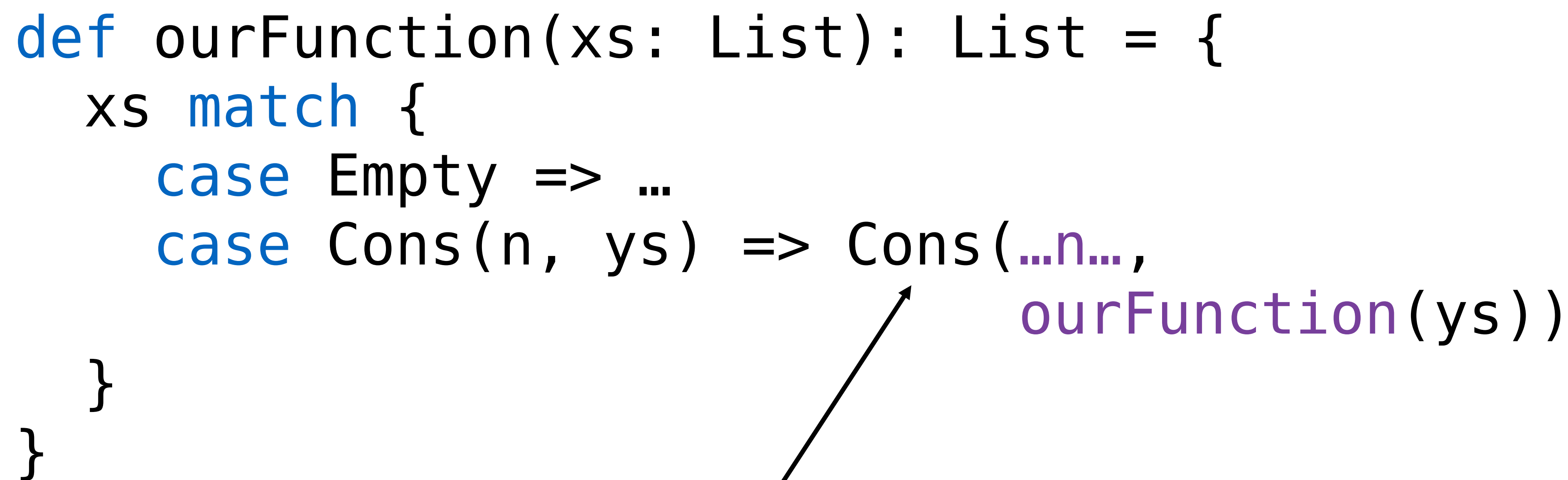
Problem Statement: Given a list of times measured in hours, we want to construct a list of corresponding times measured in seconds

Converting Hours to Seconds

```
def hoursToSeconds(xs: List): List = {
  xs match {
    case Empty => Empty
    case Cons(n, ys) => Cons(seconds(n), hoursToSeconds(ys))
  }
}

def seconds(hours: Int) = 3600 * hours
```

Generalizing to a Template



```
def ourFunction(xs: List): List = {
  xs match {
    case Empty => …
    case Cons(n, ys) => Cons(…n…,
                             ourFunction(ys))
  }
}
```

Really, this is the same template as before, but now Cons is our combining operation

The Natural Numbers

```
Nat ::= 0
      | Next(Nat)
```

The Natural Numbers

```
Nat ::= 0
      | Next(Nat)
```

Here we are between Cases One and Two for Abstract Datatypes:

- No new variants expected
- Many new functions expected
- But some basic functions are intrinsic to the type

Defining The Natural Numbers in Scala

```scala
sealed abstract class Nat
case object Zero extends Nat
case class Next(n: Nat) extends Nat
```

Defining The Natural Numbers in Scala

```
sealed abstract class Nat {
  def +(n: Nat): Nat
  def *(n: Nat): Nat
}
```

Defining The Natural Numbers in Scala

```
case object Zero extends Nat {
  def +(n: Nat) = n
  def *(n: Nat) = Zero
}

case class Next(n: Nat) extends Nat {
  def +(m: Nat) = Next(n + m)
  def *(m: Nat) = m + (n * m)
}
```

Defining The Natural Numbers in Scala

```
case object Zero extends Nat {
  def +(n: Nat) = n
  def *(n: Nat) = Zero
}

case class Next(n: Nat) extends Nat {
  def +(m: Nat) = Next(n + m)
  def *(m: Nat) = m + (n * m)
}
```

Again we have natural recursion: base case, recursion, combination

Example Reduction
(3 + 2)

```
Next(Next(Next(Zero)) + Next(Next(Zero))  ↦
Next(Next(Next(Zero)) + Next(Next(Zero))) ↦
Next(Next(Next(Zero) + Next(Next(Zero)))) ↦
Next(Next(Next(Zero + Next(Next(Zero))))) ↦
     Next(Next(Next(Next(Next(Zero)))))
```

Factorial

```
def factorial(n: Nat): Nat = {
  n match {
    case Zero => Next(Zero)
    case Next(m) => n * factorial(m)
  }
}
```

Transferring The Pattern To Ints

```
def factorial(n: Int): Int = {
  require (n >= 0)

  if (n == 0) 1
  else n * factorial(n - 1)

} ensuring (_ > 0)
```

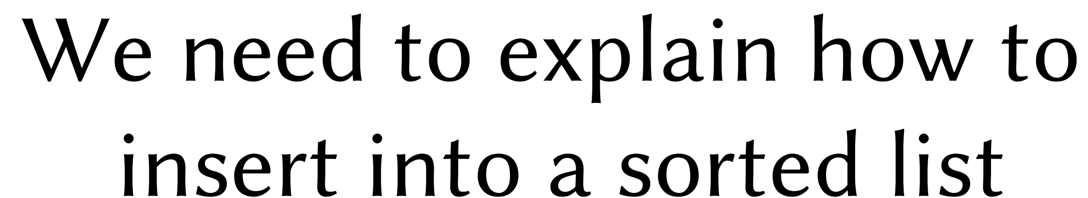
Combining Via Auxiliary Functions

Combining Via Auxiliary Functions

- As our examples with natural numbers shows, it is often desirable to define the *combining operation* of a natural recursion as an auxiliary function
- We can apply this insight to lists and use our template to cover yet more cases

Sorting Lists

```scala
def sort(xs: List): List = {
  xs match {
    case Empty => Empty
    case Cons(n, ys) => insert(n, sort(ys))
  }
}
```



Insertion

```
def insert(n: Int, xs: List): List = {
  xs match {
    case Empty => Cons(n, Empty)
    case Cons(m, ys) => {
      if (n <= m) Cons(n, xs)
      else Cons(m, insert(n, ys))
    }
  }
}
```

Insertion

```
def insert(n: Int, xs: List): List = {
  xs match {
    case Empty => Cons(n, Empty)
    case Cons(m, ys) => {
      if (n <= m) Cons(n, xs)
      else Cons(m, insert(n, ys))
    }
  }
}
```

This parameter is not traversed,
but is used for combination and comparison
Other functions follow this pattern.

Appending Two Lists

```
sealed abstract class List {
  /**
   * Returns a new list with the elements of
   * this list appended to the given list.
   */
  def ++(ys: List): List
}
```

Appending Two Lists

```
case object Empty extends List {
  def ++(ys: List) = ys
}
```

Appending Two Lists

```
case class Cons(first: Int, rest: List) extends List {
  def ++(ys: List) = Cons(first, rest ++ ys)
}
```

Recursively Defined Datatypes with Multiple Recursive Parameters

Family Trees

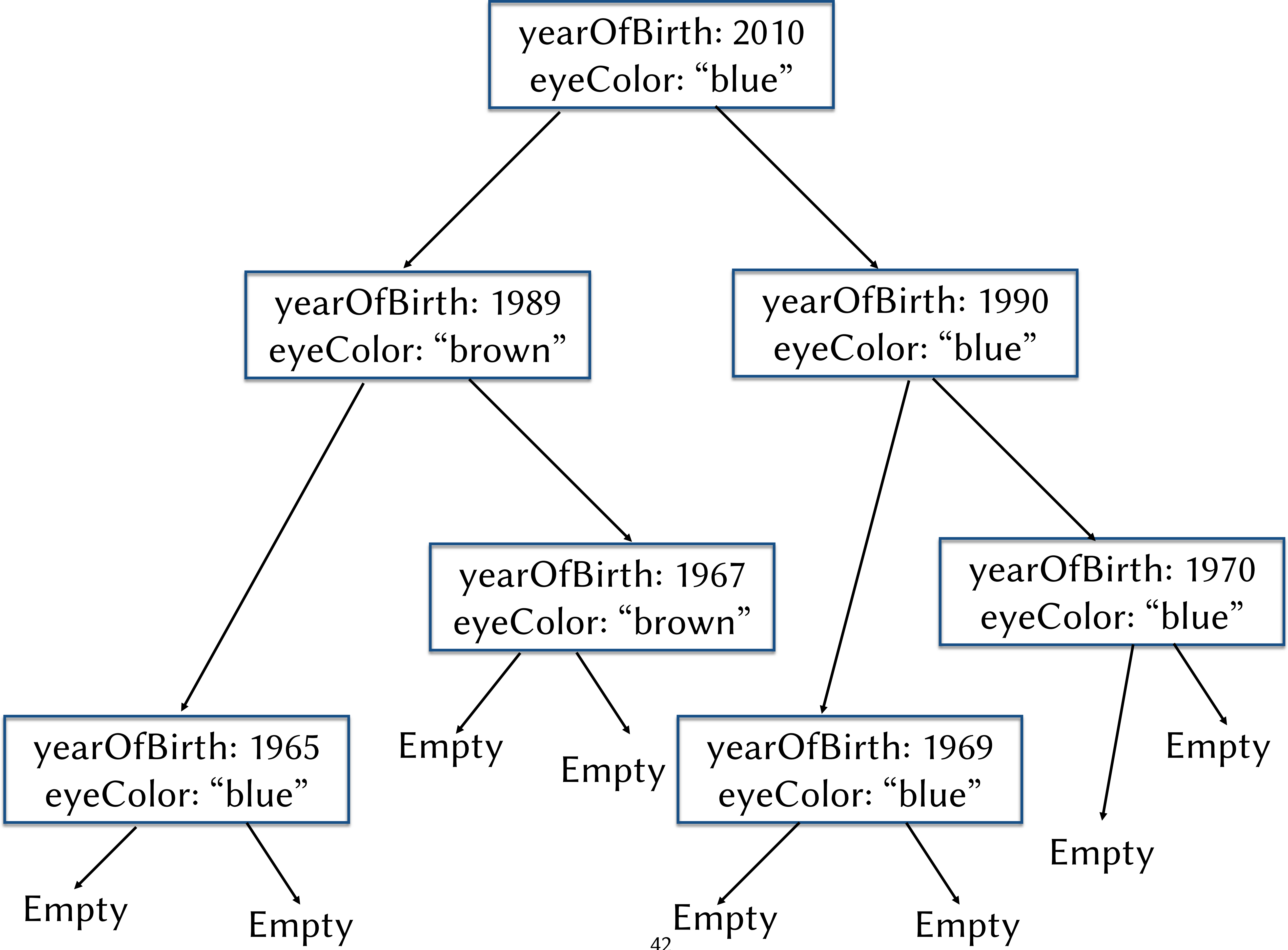
```
TreeNode ::= Empty
           | Child(TreeNode,
                   TreeNode,
                   Int,
                   String)
```

Family Trees

```scala
sealed abstract class TreeNode

case object EmptyNode extends TreeNode

case class Child(
    mother: TreeNode,
    father: TreeNode,
    yearOfBirth: Int,
    eyeColor: String)
  extends TreeNode
```


yearOfBirth: 2010
eyeColor: “blue”
yearOfBirth: 1989
eyeColor: “brown”
yearOfBirth: 1990
eyeColor: “blue”
yearOfBirth: 1967
eyeColor: “brown”
yearOfBirth: 1970
eyeColor: “blue”
yearOfBirth: 1965
eyeColor: “blue”
Empty
Empty
yearOfBirth: 1969
eyeColor: “blue”
Empty
Empty
Empty
Empty
Empty
Empty

Family Trees

```
def hasBlueEyedAncestor(t: TreeNode): Boolean =
  t match {
    case EmptyNode => false
    case Child(mother,father,_,eyeColor) =>
      ( (eyeColor == "Blue")
      || hasBlueEyedAncestor(mother)
      || hasBlueEyedAncestor(father) )
  }
```

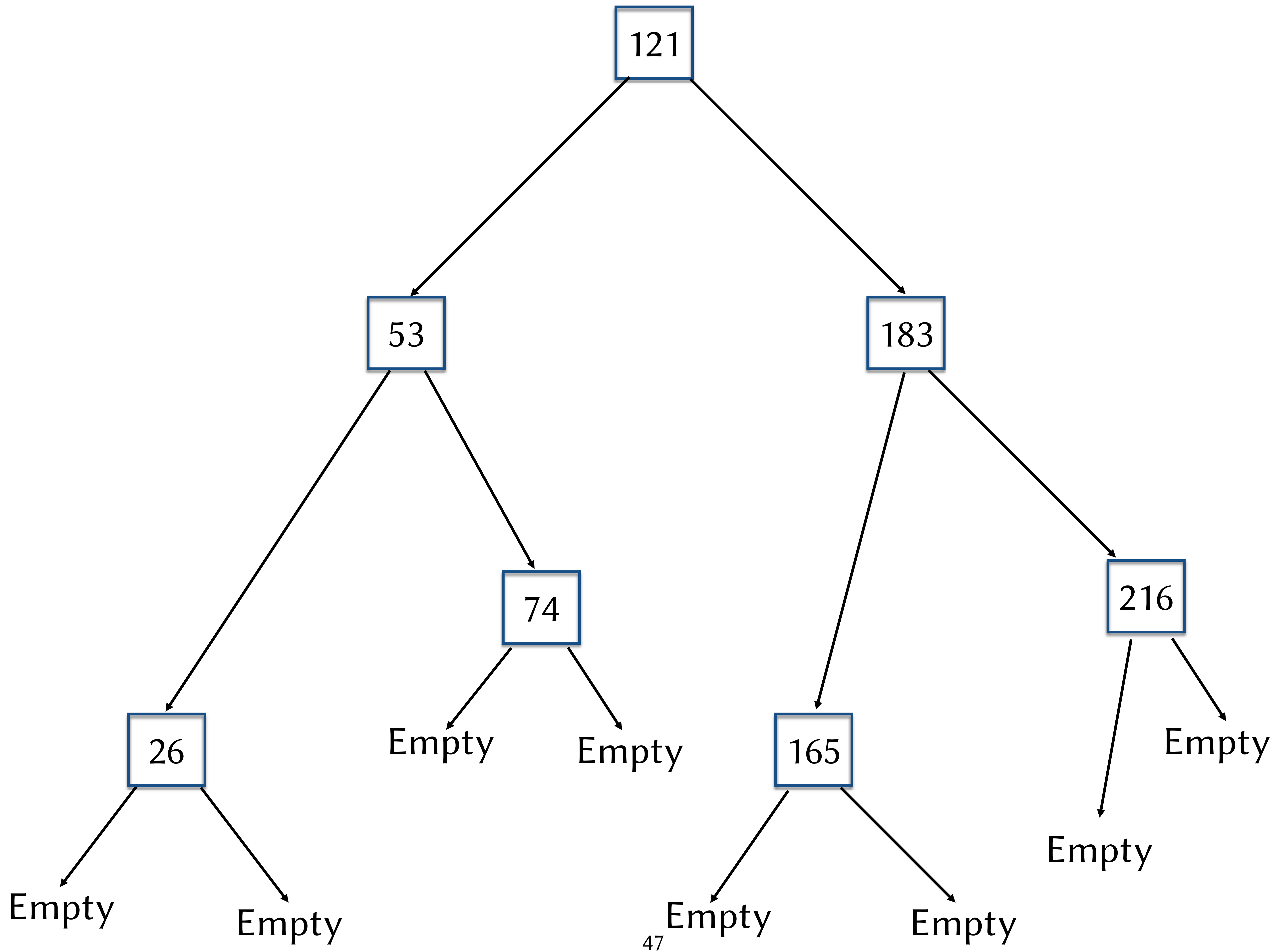
Family Trees

```
def hasBlueEyedAncestor(t: TreeNode): Boolean =
  t match {
    case EmptyNode => false
    case Child(_,_,_,"Blue") => true
    case Child(mother,father,_,_) =>
      hasBlueEyedAncestor(mother) ||
      hasBlueEyedAncestor(father)
  }
```

Binary Search Trees

Binary Search Trees

- We define trees containing only Ints
- To help us find elements quickly, we abide by the following invariant:
  - At a given node containing value *n*:
    - All values in the left subtree are less than *n*
    - All values in the right subtree are greater than *n*


121
53
183
26
74
165
216
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty

Binary Search Trees

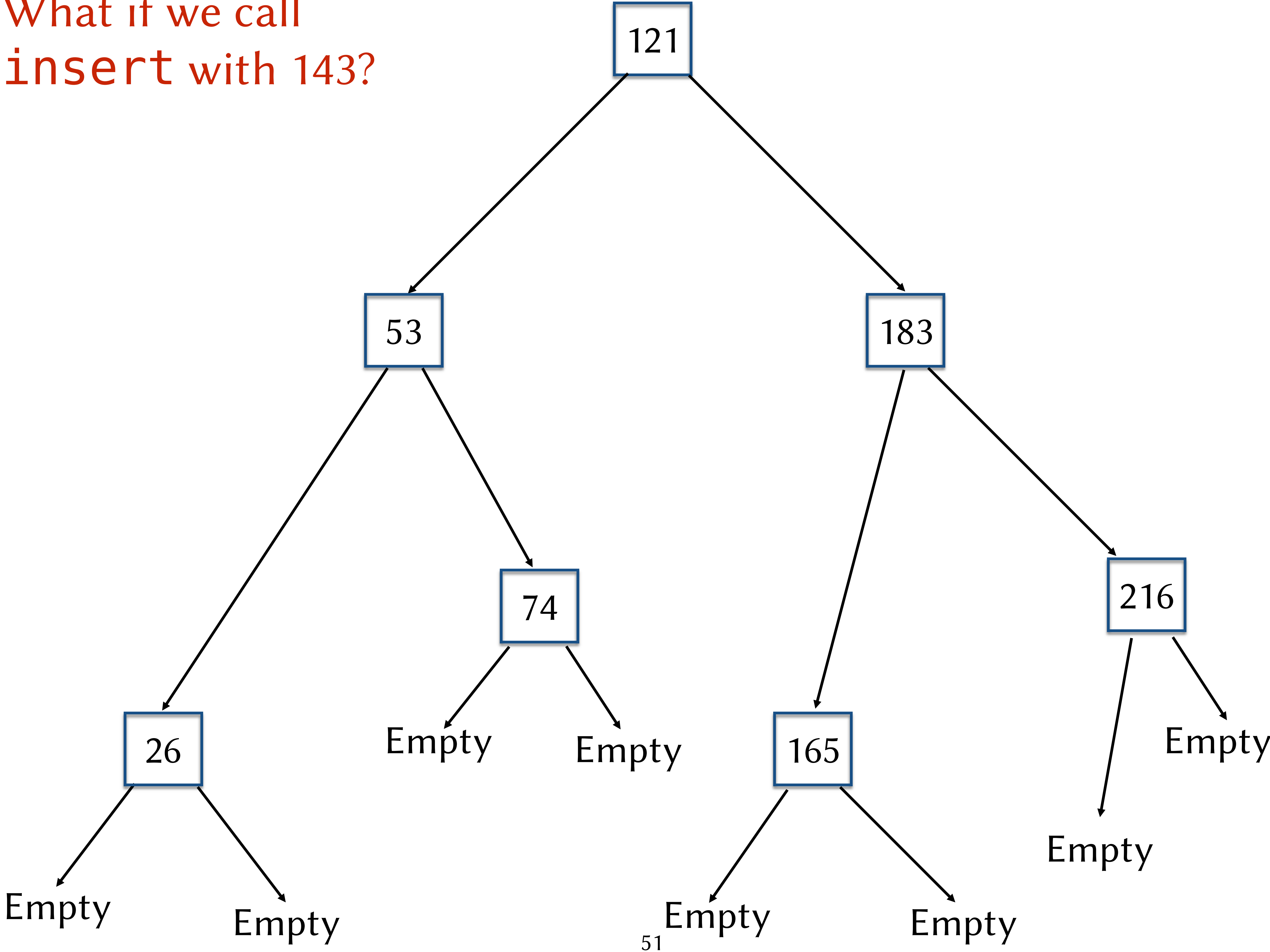
```
abstract class BinarySearchTree {
  def contains(n: Int): Boolean
  def insert(n: Int): BinarySearchTree
}
```

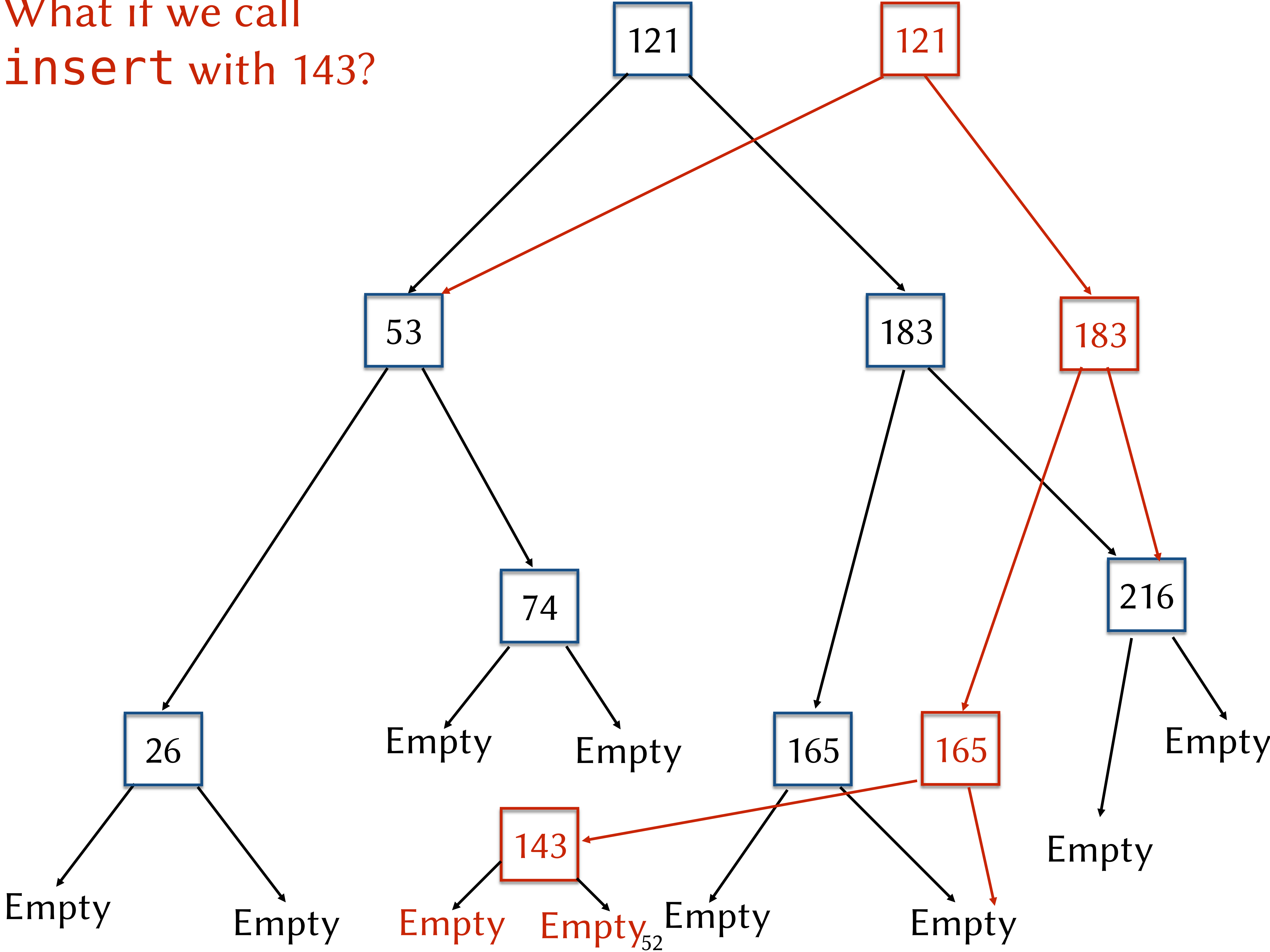
Binary Search Trees

```
case object EmptyTree extends BinarySearchTree {
  def contains(n: Int) = false
  def insert(n: Int) = ConsTree(n, EmptyTree, EmptyTree)
}
```

Binary Search Trees

```
case class ConsTree(
    m: Int,
    left: BinarySearchTree,
    right: BinarySearchTree)
  extends BinarySearchTree {
  def contains(n: Int): Boolean = {
    if (n < m) left.contains(n)
    else if (n > m) right.contains(n)
    else true // n == m
  }
  def insert(n: Int) = {
    if (n < m) ConsTree(m, left.insert(n), right)
    else if (n > m) ConsTree(m, left, right.insert(n))
    else this // n == m
  }
}
```


What if we call insert with 143?
121
53
183
26
74
165
216
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty


What if we call insert with 143?
121
121
53
183
183
26
74
216
165
165
143
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty
Empty
52

Taking the First Few Elements

```
def take(n: Int, xs: List): List = {
  require(0 <= n && n <= xs.size)
  (n, xs) match {
    case (0, xs) => Empty
    case (n, Cons(y, ys)) => Cons(y, take(n-1, ys))
  }
}
```

Dropping the First Few Elements

```
def drop(n: Int, xs: List): List = {
  require(0 <= n && n <= xs.size)
  (n, xs) match {
    case (0, xs) => xs
    case (n, Cons(y, ys)) => drop(n-1, ys)
  }
}
```

Functional Update of a List

```
def update(xs: List, i: Int, y: Int): List = {
  require(0 <= i && i < xs.size)
  assume(xs != Empty) // implied by requirements

  (xs, i) match {
    case (Cons(z, zs), 0) => Cons(y, zs)
    case (Cons(z, zs), _) => Cons(z, update(zs,i-1,y))
  }
}
```