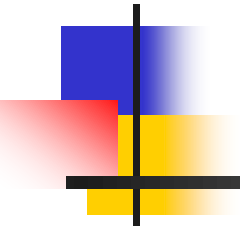

Lambda the Ultimate and Reduction Semantics

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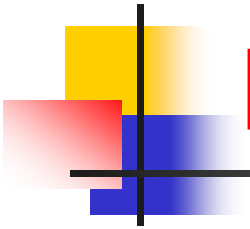
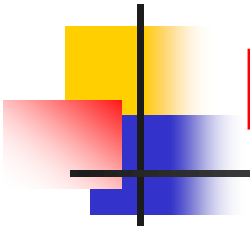


Functions Are Values

- In most mainstream languages, functions are not “first-class” values. In contrast, in almost every functional language, functions are first-class data values. They can be
  - bound to variables (including function parameters);
  - returned as results from functions;
  - dynamically constructed during computation just like ordinary data values, but this process is clumsy without better notation that we have used so far.
- Terminology
  - A functional language where functions are not values is called a *first-order* functional language. SISAL is such a language. Java is often called a first-order language because methods are not data values.
  - A functional language where functions are regular data values (as described above) is called a *higher-order* functional language. Nearly all modern functional languages are higher-order.



Motivation for λ-notation

- Many functions in programs are used only once.
  - Examples: arguments to functions like
    - **`map`**,
    - **`filter`**,
    - **`fold`**, and many more "higher-order" functions (functions that take functions as arguments)
- Sometimes we want to build new functions in the middle of a computation. The **`local`** construct suffices but it is notationally clumsy for this purpose.
- λ provides simpler, more concise notation
- Invented by Alonzo Church in the 1930s

Basic Idea

- λ-notation was invented by mathematicians (Church *et al*). For example, given

  f(x) = x² + 1

  what is **f**? **f** is the function that maps **x** to **x² + 1** which we might write as

  x → x² + 1

  The latter avoids naming the function. The notation

  λx . x² + 1 evolved instead of **x → x² + 1** (for type-setting convenience?

- In Scheme, we write `(lambda (x) (+ (* x x) 1))` instead of **λx . x² + 1**.

- `(define (f x) (+ (* x x) 1))` abbreviates

  `(define f (lambda (x) (+ (* x x) 1)))`

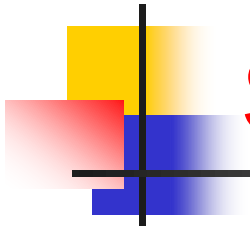
Why λ?

- The name was used by its inventor
  - Alonzo Church, logician, 1903-1995.
  - Princeton, NJ
  - Introduced lambda in 1930's to formalize mathematical proofs
- Church is my academic great-grandfather

  Alonzo Church -> Hartley Rogers ->

  David Luckham -> Corky Cartwright

Scope for a Lambda Abstraction



- Argument scope: **(lambda (**x_1 **...** x_n**)** ***body*****)** introduces the variables x_1 **...** x_n which have *body* as their scope (except for holes)
- Example:

  (lambda (x) (+ (* x x) 1))
- Scope for variable introduced by **define**.
  - At the top-level,

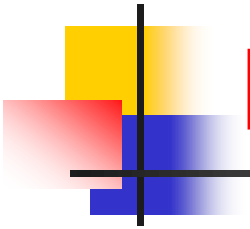
    (define f ***rhs*****)**

    introduces the variable **f** which is visible everywhere (except inside holes introduced by local definitions of **f**, including **lambda** bindings .
  - Inside the expression

    (local [(define f_1 rhs_1**) ... (define** f_n rhs_n**))** ***body*****)**
  - the variables f_1 **...** f_n have the entire **local** expression as their scope.
- Recursion comes from **define** not **lambda**! It is possible to define recursive functions solely using **lambda** (and whatever primitive operations exist in the language) but it is surprisingly hard. The solution is called the Y-operator.

Some PL researchers are crazy about λ!

Prof.
Phil Wadler
University of
Edinburgh



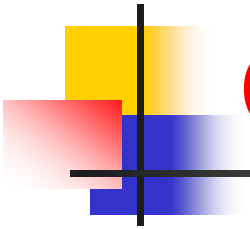
Example

By using embedded lambda-abstraction, we can write the following program concisely

```
(define l1 '(1 2 3 4 5))
(define l2
  (local ((define (square x) (* x x)))
    (map square l1)))
```

as

```
(define l1 '(1 2 3 4 5))
(define l2 (map (lambda (x) (* x x)) l1))
```



Careful Definition of Syntax

- Official specification of what expressions that use `lambda` look like:
  - ***exp*** **= ... | (lambda (*var**) *exp*)**
- Interesting points
  - In Racket/Scheme, may have multiple arguments
  - May have have no arguments
- Application of a function with no arguments

```
(define blowup (lambda () (/ 1 0)))
(blowup)
```

Currying: An Important Formulation of Lambda Notation

- In the original formulation of the Lambda Calculus, lambda-abstraction was limited to a single argument because every function

  $f: D_1 \times \ldots \times D_n \rightarrow D$

  is isomorphic to a function

  $f_c: D_1 \rightarrow \ldots \rightarrow D_n \rightarrow D$

  eliminating the need for abstractions involving more than one variable. The function f_c is called the "curried" equivalent of *f*, in honor of Haskell Curry, who was prominent among the small group of logicians who formalized a system of mathematical notation starting in the 1920s, called *combinatory logic*, in which functions are ordinary values. The originator of the idea as recorded in the published literature was Moses Schönfinkel, a Russian logician and a colleague of Hilbert who warrants more fame. (Perhaps "schönfinkeling" is too much of a mouthful.)

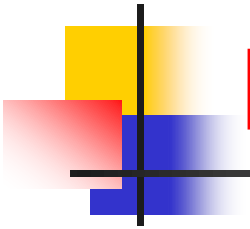
  Ironically, purely combinatory languages do not include lambda abstraction because they dispense with variables. This is a convenient and common "mathematical hack" in formal logic. Combinatory languages are not very intuitive. John Backus (the inventor of Fortran) developed a purely combinatory language called FP but it included a syntactic hack to make code more readable. I generalized essentially the same hack in my paper *Lambda as a Combinator*, published in the Festschrift for John McCarthy.

Currying: An Important Formulation of Lambda Notation cont.

In contrast Racket supports multiple arguments including the option of no arguments. In statically typed functional languages (like Ocaml and Haskell) lambda-abstraction is typically limited to a single argument implying that all functions are "curried". In practice, this convention has proven very convenient provided the parentheses around the argument in a function application are dropped. Infix notation can be supported as alternate notation for the full application of a curried binary function. Check out the Haskell language at haskell.org. (Wisecrack: perhaps it should called "Moses" instead of "Haskell".)

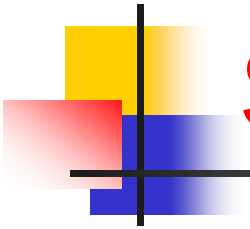
A simple example of the curried formulation of a familiar function is curried addition. In Racket, it can be defined by the lambda abstraction

```
(lambda (x) (lambda (y) (+ x y)))
```



Reduction Semantics

- Simple Reduction Semantics: Essence of Functional Programming
- Idea: Evaluation of expressions is a familiar idea from grammar school.
- Grammar school:
  evaluate parenthesized arithmetic expressions
- Functional programming:
  evaluate arbitrary (functional program) text
- Only significant difference: in reduction semantics, reduction order is unique (required for determinism)



Synopsis of Reduction

- Value are values are values …
- A value evaluates to itself so we stop evaluation when we reduce our original expression to a value.
- In most functional languages, always perform leftmost reductions because order matters

Evaluation of λ-expressions

- How do we evaluate a λ-abstraction

  (lambda (x$_1$ ... x$_n$) *body*)

  It's a value!

- What about λ-applications?

  ((lambda (x$_1$ ... x$_n$) *body*) v$_1$ ... v$_n$)

  => *body*[x$_1$←v$_1$, ..., x$_n$←v$_n$] (called β-reduction)

  Examples:

```
((lambda (x) (* x 5)) 4) => (* 4 5) => 20

((lambda (x) (x x)) (lambda (x) (x x)))
=> ((lambda (x) (x x)) (lambda (x) (x x)))
=> ((lambda (x) (x x)) (lambda (x) (x x)))
```

  Note: ***body***[x$_1$←v$_1$, ..., x$_n$←v$_n$] means ***body*** with v$_1$ substituted for x$_1$, ..., v$_n$ substituted for x$_n$.

Problem with Raw Substitution

The reduction

```
((lambda (x) (lambda (y) (y x)))
  (lambda (z) (+ y z)))

=> (lambda (y) (y (lambda (z) (+ y z)))))
```

WRONG!

The meaning of **y** has changed! But this pathology (called the *capture* of **y**) can *never* happen in the evaluation of Racket program text using our evaluation rules *provided* **`lambda`** *is the only binding construct*. A functional argument like

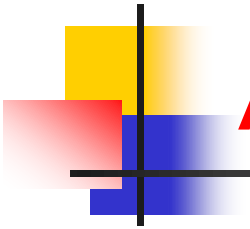
```
(lambda (z) (+ y z)))
```

with a free variable (**y** in our example) never appears as a value in a reduction *unless the free variable is defined in a top-level* **`define`**. HW3 explores this issue in Problem 5.

Preventing Capture: Safe Substitution

We can avoid capturing a free variable (like `y` in our example) by renaming local variables in the code body that would otherwise capture free variables in the argument expression that is being substituted. Consider the Racket program

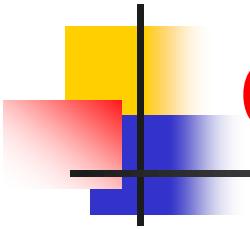
```
   (define y 5)
   ((lambda (f) ((lambda (y) (f y)) 10)) (lambda (z) (+ y z)))
=> ...  [rename y to avoid capturing y;  this step is typically combined with the next step]
   ((lambda (f) ((lambda (x) (f x)) 10)) (lambda (z) (+ y z)))
=> ...
   ((lambda (x) ((lambda (z) (+ y z)) 10))
=> ...
   ((lambda (z) (+ y z)) 10))
=> ...
   (+ y 10)
=> ...
   15
```



Avoiding Safe Substitution

Safe substitution is not mentioned in the evaluation rules given in HTDP. As a result, the HTDP rules fail in pathological examples like the one given on the previous slide.

Nevertheless, the DrRacket stepper computes the correct answer because it distinguishes top-level usage occurrences (like **y** in the example on the preceding slide) from embedded usage occurrences (like **y** in our example). This distinction is ignored in the evaluation rules given in HTDP, which only identifies binding and usage occurrences of variables. We can fix this minor error in the HTDP evaluation rules either by using "safe substitution" in the beta-reduction rule (as in this class) or by introducing two different forms of variable usage occurrences: references to embedded variables (introduced in **`lambda`** and local **`define`** bindings). The DrRacket stepper relies on the second fix.



Comprehensive Reduction Rules

- The document Laws Of Evaluation entitled ***Evaluating Core Racket Programs*** is a comprehensive description of the reduction semantics of functional Racket. In the literature, this form of semantics is often called a "rewrite-rule" semantics. A similar system, with minor technical differences, called Structural Operational Semantics, is widely cited by computer science researchers (see Wikipedia!) but regrettably is almost completely ignored in the documentation of mainstream languages.
- As you see in HW3, you need to understand the rules in detail to understand the corresponding programs.