# COMP 322: Fundamentals of Parallel Programming

## Lecture 34: Algorithms based on Parallel Prefix (Scan) operations

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# Beyond Sum/Reduce Operations - Prefix Sum (Scan)

Given input array A, compute output array X as follows: $$X[i] = \sum_{0 \le j \le i} A[j]$$

- The above is an inclusive prefix sum since X[i] includes A[i]
- For an exclusive prefix sum, perform the summation for 0 <=j <i
- It is easy to see that inclusive prefix sums can be computed sequentially in O(n) time …

```
// Copy input array A into output array X
X = new int[A.length]; System.arraycopy(A,0,X,0,A.length);
// Update array X with prefix sums
for (int i=1 ; i < X.length ; i++ ) X[i] += X[i-1];
```

- … and so can exclusive prefix sums

# Summary of Parallel Prefix Sum Algorithm (Recap - Lecture 13)

- Critical path length, CPL = O(log n)
- Total number of add operations, WORK = O(n)
- Optimal algorithm for P = O(n/log n) processors
  —Adding more processors does not help
- Parallel Prefix Sum has several applications that go beyond computing the sum of array elements
  - Parallel Prefix Sum can be used for any operation that is associative (need not be commutative)
    —In contrast, finish accumulators required the operator to be both associative and commutative



# Parallel Filter Operation (Recap)

[Credits: David Walker and Andrew W. Appel (Princeton), Dan Grossman (U. Washington)]

Given an array `input`, produce an array `output` containing only elements such that `f(elt)` is `true`, i.e., output = input.parallelStream.filter(f).toArray

Example:
```
input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
f: is elt > 10
output [17, 11, 13, 19, 24]
```

Parallelizable?

—Finding elements for the output is easy

—But getting them in the right place seems hard



# Parallel Prefix to the rescue (Recap)

1. Parallel map to compute a bit-vector for true elements (can use Java streams)

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector (not available in Java streams)

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. Parallel map to produce the output (can use Java streams)

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){
  if(bits[i]==1)
    output[bitsum[i]-1] = input[i];
}
```



# Example Applications of Parallel Prefix Algorithm

- Prefix Max: given an input array A, output an array X of objects such that X[i].max is the maximum of elements A[0…i]

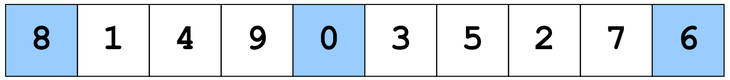
- Filter and Packing of Strings: given an input array A identify elements that satisfy some desired property (e.g., uppercase), and pack them in a new output array. (First create a 0/1 array for elements that satisfy the property, and then compute prefix sums to identify locations of elements to be packed.)

  —Useful for parallelizing partitioning step in Parallel Quicksort algorithm

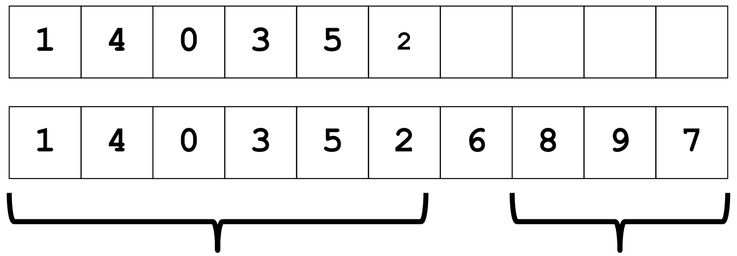


# Parallelizing Quicksort Example

- Step 1: pick pivot as median of three




- Steps 2: implement partition step as two filter/pack operations that store result in a second array




- Step 3: Two recursive sorts in parallel

# Use of Prefix Sums to Parallelize partition() in Quicksort

```
partition(int[] A, int M, int N) {
 pivot = … ; // choose pivot from M..N
 Allocate temporary buffer[] with size N-M+1 elements
 forall (point [k] : [0:N-M]) { // parallel loop
  lt[k] = (A[M+k] < A[pivot] ? 1 : 0);  // bit vector with < comparisons
  eq[k] = (A[M+k] == A[pivot] ? 1 : 0); // bit vector with = comparisons
  gt[k] = (A[M+k] > A[pivot] ? 1 : 0);  // bit vector with > comparisons
  buffer[k] = A[M+k];                   // Copy A[M..N] into buffer
 }
 // computePrefixSums() returns the prefix sum array and the total count of 1's in the input array
ltPs, ltCount = computePrefixSums(lt);
eqPs, eqCount = computePrefixSums(eq);
fgtPs, gtCount = computePrefixSums(gt);
// Parallel move from buffer into A
forall (point [k] : [0:N-M]) {
  if(lt[k]==1) A[M+ltPS[k]-1] = buffer[k];
  else if(eq[k]==1) A[M+ltCount+eqPS[k]-1] = buffer[k];
  else A[M+ltCount+eqCount+gtPS[k]-1] = buffer[k];
 }
}  // partition
```



# Formalizing Parallel Prefix: Scan and Pre-scan operations

- The *scan* operation is an inclusive parallel prefix sum operation.

- The scan operator was introduced in APL in the 1960's, and has been popularized recently in more modern languages, most notably the NESL project in CMU



# Formalizing Parallel Prefix: Scan and Pre-scan operations

- The *prescan* operation is an exclusive parallel prefix sum operation. It takes a binary associative operator $\oplus$ with identity I, and a vector of n elements, [a0, a1, ..., an−1], and returns the vector [I,a0,(a0 $\oplus$a1),...,(a0 $\oplus$a1 $\oplus$…$\oplus$an−2)].

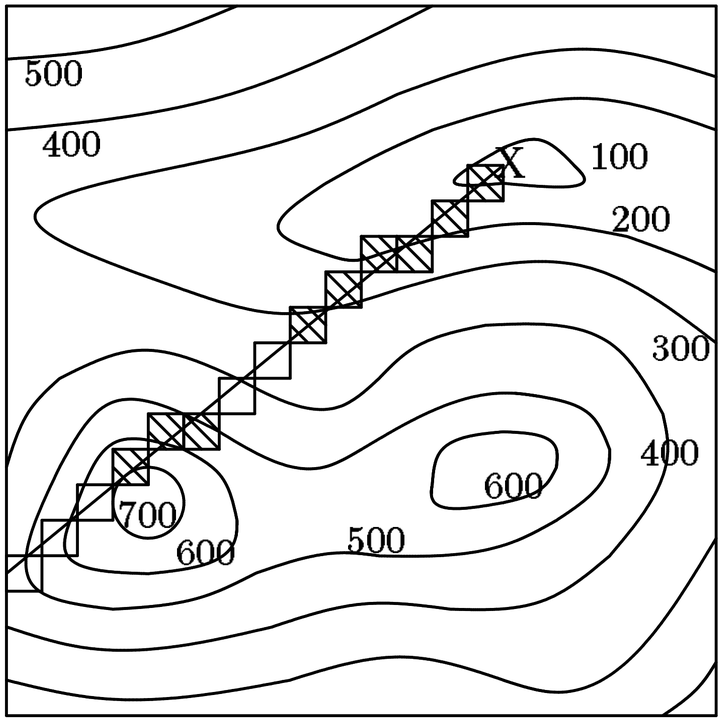
- A prescan can be generated from a scan by shifting the vector right by one and inserting the identity. Similarly, the scan can be generated from the prescan by shifting left, and inserting at the end the sum of the last element of the prescan and the last element of the original vector.



# Line-of-Sight Problem

- Problem Statement: given a terrain map in the form of a grid of altitudes and an observation point, X, on the grid, find which points are visible along a ray originating at the observation point. Note that a point on a ray is visible if and only if no other point between it and the observation point has a greater vertical angle.

- Define angle[i] = angle of point i on ray relative to observation point, X (can be computed from altitudes of X and i)
- A max-prescan on angle[*] returns to each point the maximum previous angle.
- Each point can compare its angle with its max-prescan value to determine if it will be visible or not



# Segmented Scan

Goal: Given a data vector and a flag vector as inputs, compute independent scans on segments of the data vector specified by the flag vector.

$$x_i = \begin{cases} a_0 & i = 0 \\ \begin{cases} a_i & f_i = 1 \\ (x_{i-1} \oplus a_i) & f_i = 0 \end{cases} & 0 < i < n \end{cases}$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a$ | = | [5 | 1 | 3 | 4 | 3 | 9 | 2 | 6] |
| $f$ | = | [1 | 0 | 1 | 0 | 0 | 0 | 1 | 0] |
| segmented +-scan | = | [5 | 6 | 3 | 7 | 10 | 19 | 2 | 8] |
| segmented max-scan | = | [5 | 5 | 3 | 4 | 4 | 9 | 2 | 6] |

# Using Segmented Scan for Quicksort

```
procedure quicksort(keys)
  seg-flags[0] ← 1
  while not-sorted(keys)
    pivots    ← seg-copy(keys, seg-flags)
    f         ← pivots <=> keys
    keys      ← seg-split(keys, f, seg-flags)
    seg-flags ← new-seg-flags(keys, pivots, seg-flags)
```

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Key | = | [6.4 | 9.2 | 3.4 | 1.6 | 8.7 | 4.1 | 9.2 | 3.4] |
| Seg-Flags | = | [1 | 0 | 0 | 0 | 0 | 0 | 0 | 0] |
| Pivots | = | [6.4 | 6.4 | 6.4 | 6.4 | 6.4 | 6.4 | 6.4 | 6.4] |
| F | = | [= | > | < | < | > | < | > | <] |
| Key ← split(Key, F) | = | [3.4 | 1.6 | 4.1 | 3.4 | 6.4 | 9.2 | 8.7 | 9.2] |
| Seg-Flags | = | [1 | 0 | 0 | 0 | 1 | 1 | 0 | 0] |
| Pivots | = | [3.4 | 3.4 | 3.4 | 3.4 | 6.4 | 9.2 | 9.2 | 9.2] |
| F | = | [= | < | > | = | = | = | < | =] |
| Key ← split(Key, F) | = | [1.6 | 3.4 | 3.4 | 4.1 | 6.4 | 8.7 | 9.2 | 9.2] |
| Seg-Flags | = | [1 | 1 | 0 | 1 | 1 | 1 | 1 | 0] |



# Announcements & Reminders

- Quiz for Unit 8 is due today at 11:59pm
- Lab 8 is due Monday, April 26th at 12pm



# Worksheet #34: Parallelizing the Split step in Radix Sort

**The Radix Sort algorithm loops over the bits in the binary representation of the keys, starting at the lowest bit, and executes a split operation for each bit as shown below. The split operation packs the keys with a 0 in the corresponding bit to the bottom of a vector, and packs the keys with a 1 to the top of the same vector. It maintains the order within both groups.**

**The sort works because each split operation sorts the keys with respect to the current bit and maintains the sorted order of all the lower bits. Your task is to show how the split operation (complete I-down) can be performed in parallel**

```
                                [101 111 011 001 100 010 111 010]
1.A =                           [5 7 3 1 4 2 7 2]
2.A⟨0⟩ =                         [1 1 1 1 0 0 1 0] //lowest bit
3.A←split(A,A⟨0⟩) =             [4 2 2 5 7 3 1 7]
4.A⟨1⟩ =                         [0 1 1 0 1 1 0 1] // middle bit
5.A←split(A,A⟨1⟩) =             [4 5 1 2 2 7 3 7]
6.A⟨2⟩ =                         [1 1 0 0 0 1 0 1] // highest bit
7.A←split(A,A⟨2⟩) =             [1 2 2 3 4 5 7 7]
```

```
procedure split(A, Flags)
  I-down ←
  I-up   rev(n - scan(+, rev(Flags)) // rev = reverse
  in parallel for each index i
    if (Flags[i])
      Index[i] ← I-up[i]
    else
      Index[i] ← I-down[i]
  result ← permute(A, Index)
```

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| A | = | [ 5 | 7 | 3 | 1 | 4 | 2 | 7 | 2 ] |
| Flags | = | [ 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 ] |
| I-down | = | [ 0 | 0 | 0 | 0 | 0 | 1 | 2 | 2 ] |
| I-up | = | [ 3 | 4 | 5 | 6 | 6 | 6 | 7 | 7 ] |
| Index | = | [ 3 | 4 | 5 | 6 | 0 | 1 | 7 | 2 ] |
| permute(A, Index) | = | [ 4 | 2 | 2 | 5 | 7 | 3 | 1 | 7 ] |

FIGURE 1.9
The split operation packs the elements with a 0 in the corresponding flag position to the bottom of a vector, and packs the elements with a 1 to the top of the same vector. The permute writes each element of A to the index specified by the corresponding position in Index.