COMP 322: Parallel and Concurrent Programming

Lecture 36: Parallel Prefix Sum

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Beyond Sum/Reduce Operations — Prefix Sum (Scan) Problem Statement

Given input array A, compute output array X as follows

$$X[i] = \sum_{0 \leq j \leq i} A[j]$$

- The above is an inclusive prefix sum since X[i] includes A[i]
- For an exclusive prefix sum, perform the summation for 0 <=j <i
- It is easy to see that inclusive prefix sums can be computed sequentially in O(n) time …

```
// Copy input array A into output array X
X = new int[A.length]; System.arraycopy(A,0,X,0,A.length);
// Update array X with prefix sums
for (int i=1 ; i < X.length ; i++ ) X[i] += X[i-1];
```

- … and so can exclusive prefix sums

An Inefficient Parallel Algorithm for Exclusive Prefix Sums

```
forall(0, X.length-1, (i) -> {
    // computeSum() adds A[0..i-1]
    X[i] = computeSum(A, 0, i-1);
}
```

Observations:

- Critical path length, CPL = O(log n)
- Total number of operations, WORK = $O(n^2)$
- With P = O(n) processors, the best execution time that you can achieve is T_P = max(CPL, WORK/P) = O(n), which is no better than sequential!

How can we do better?

Assume that input array A = [3, 1, 2, 0, 4, 1, 1, 3]

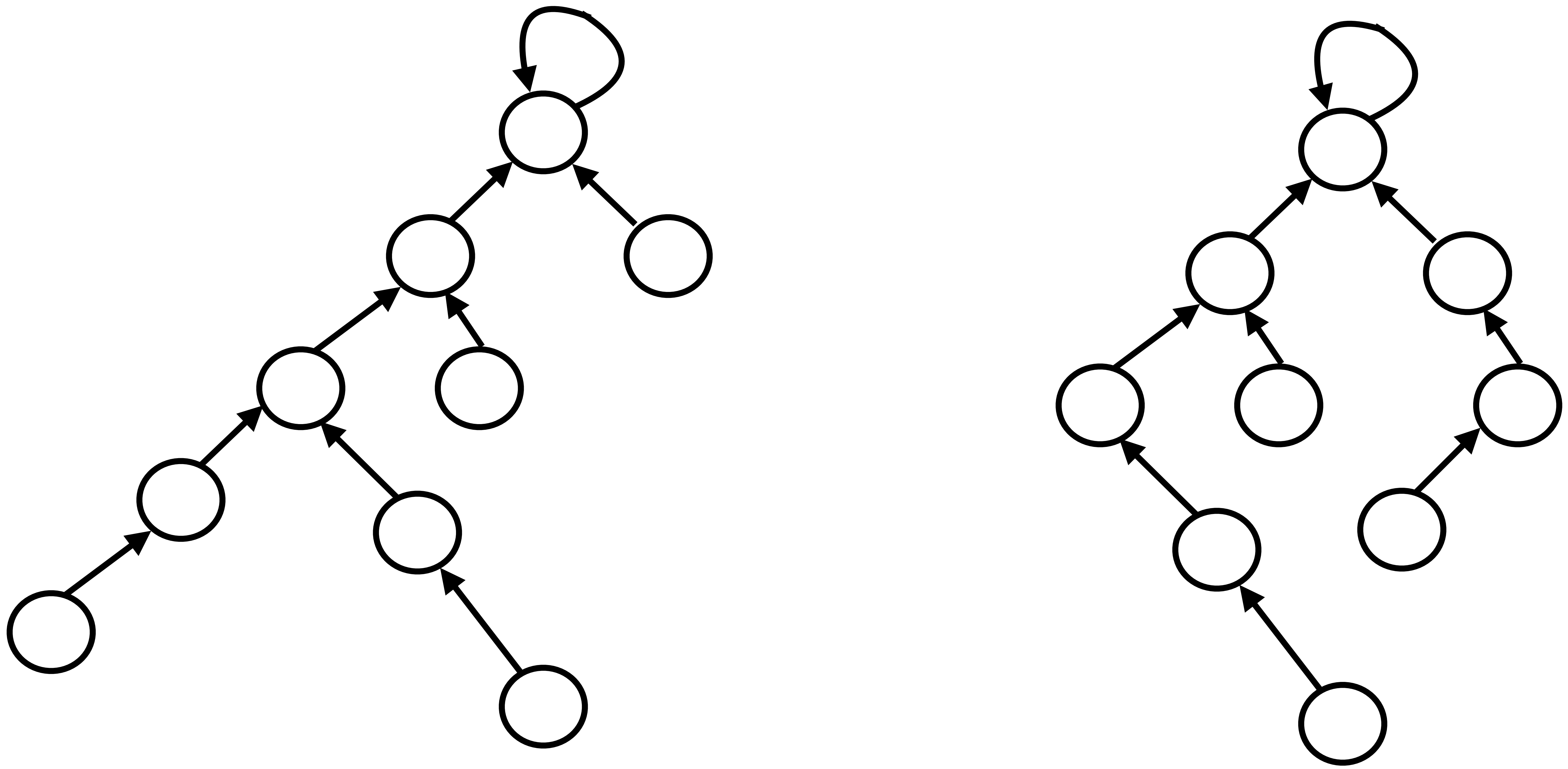
Define scan(A) = exclusive prefix sums of A = [0, 3, 4, 6, 6, 10, 11, 12]

Hint:

- Compute B by adding pairwise elements in A to get B = [4, 2, 5, 4]
- Assume that we can recursively compute scan(B) = [0, 4, 6, 11]
- How can we use A and scan(B) to get scan(A)?

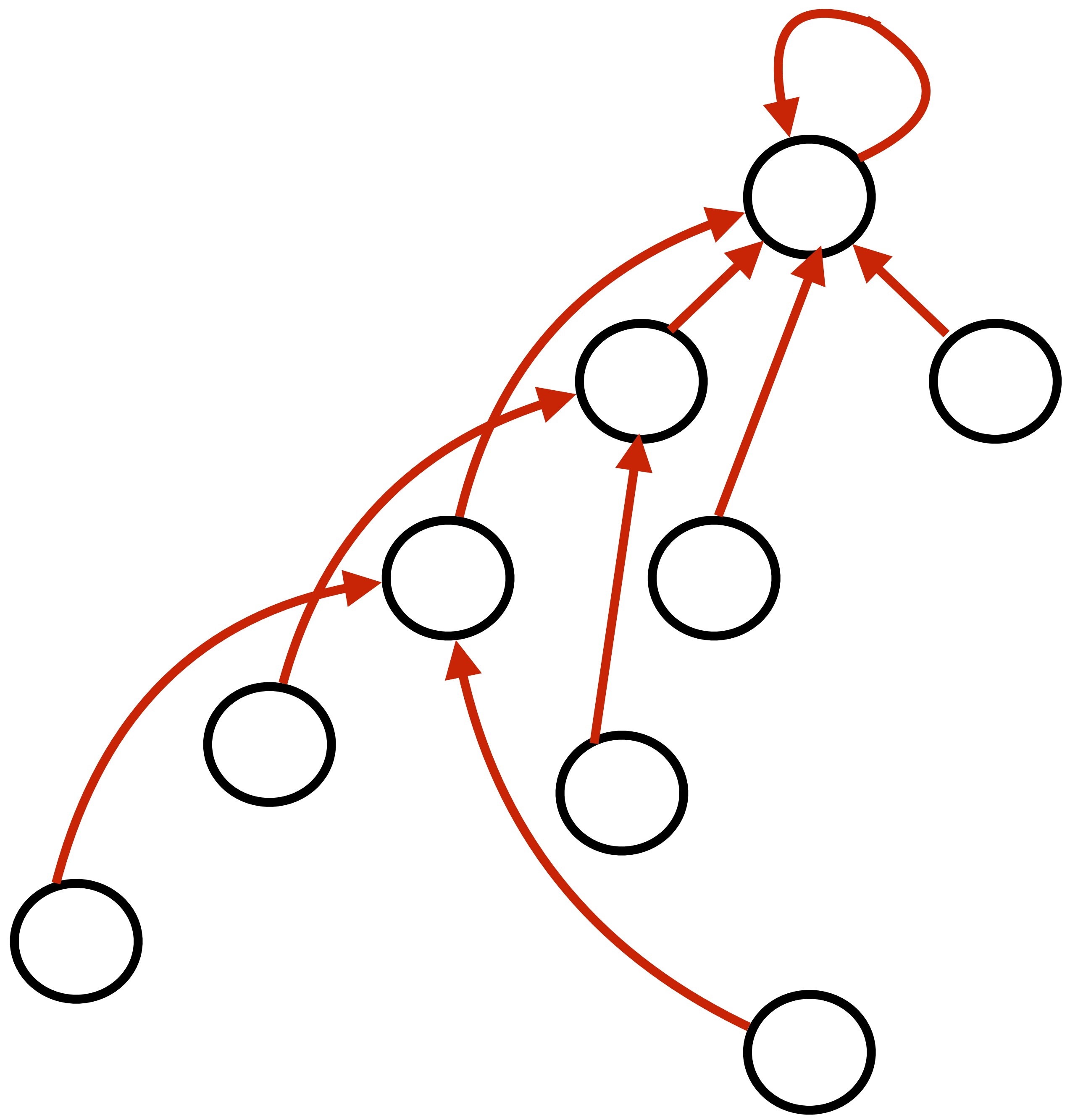
Remember the “Pointer Skipping” Idea?

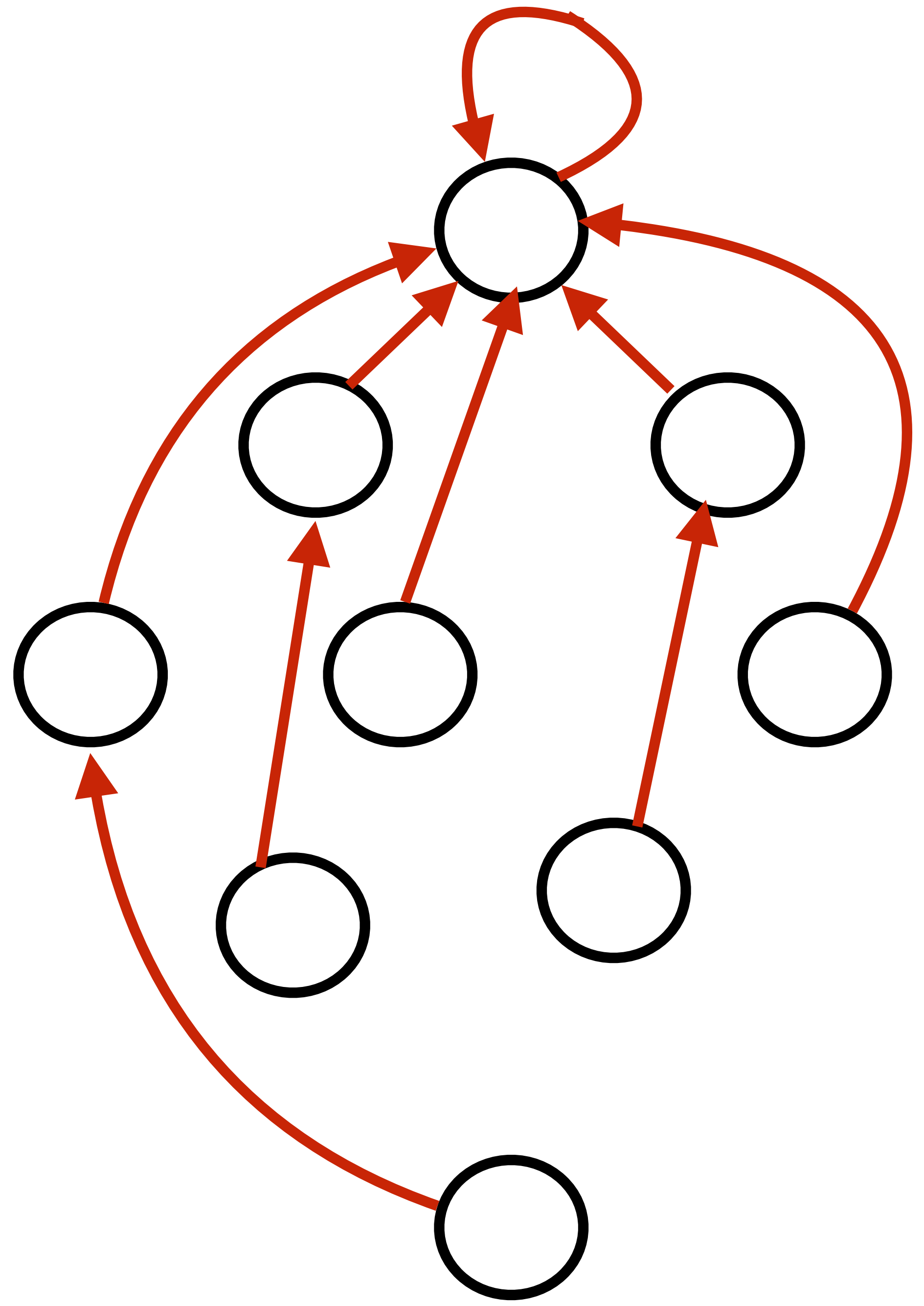
- Set each node’s root to its parent
- For each node, set its root to its parent’s root, if it exists
- This can all be done in parallel using N tasks



Remember the "Pointer Skipping" Idea?

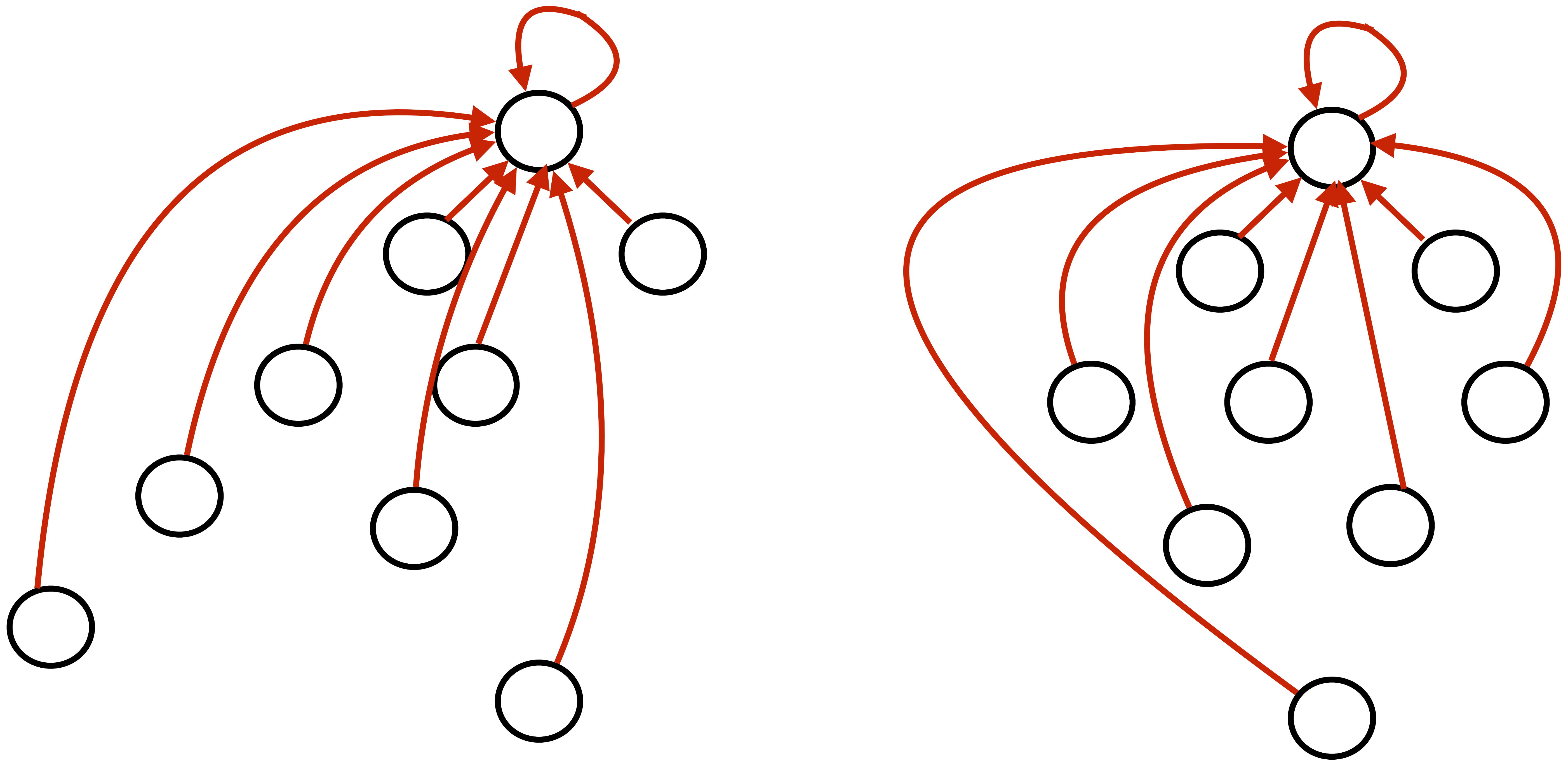
- For each node's root starts as its parent
- For each node, set its root to its parent's root, if it exists
- This can all be done in parallel using N tasks





Remember the “Pointer Skipping” Idea?

- Again:
- For each node, set its root to its parent’s root, if it exists
- This can all be done in parallel using N tasks again
- Stop when no more updates can be done



Another way of looking at the parallel algorithm

Observation: each prefix sum can be decomposed into reusable terms of power-of-2-size

e.g.

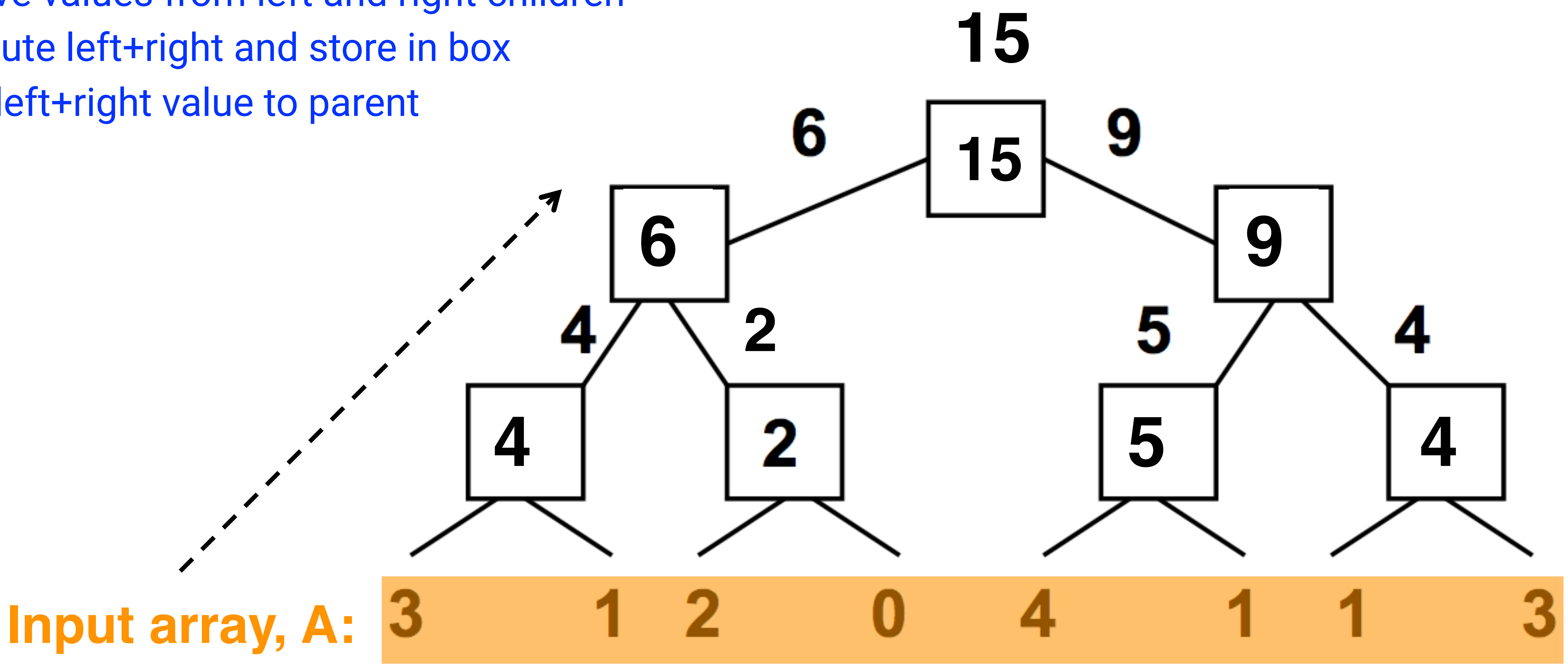
$$
\begin{aligned}
X[6] &= A[0] + A[1] + A[2] + A[3] + A[4] + A[5] + A[6] \\
&= (A[0] + A[1] + A[2] + A[3]) + (A[4] + A[5]) + A[6]
\end{aligned}
$$

Approach:

- Combine reduction tree idea from Parallel Array Sum with partial sum idea from Sequential Prefix Sum
- Use an “upward sweep” to perform parallel reduction, while storing partial sum terms in tree nodes
- Use a “downward sweep” to compute prefix sums while reusing partial sum terms stored in upward sweep

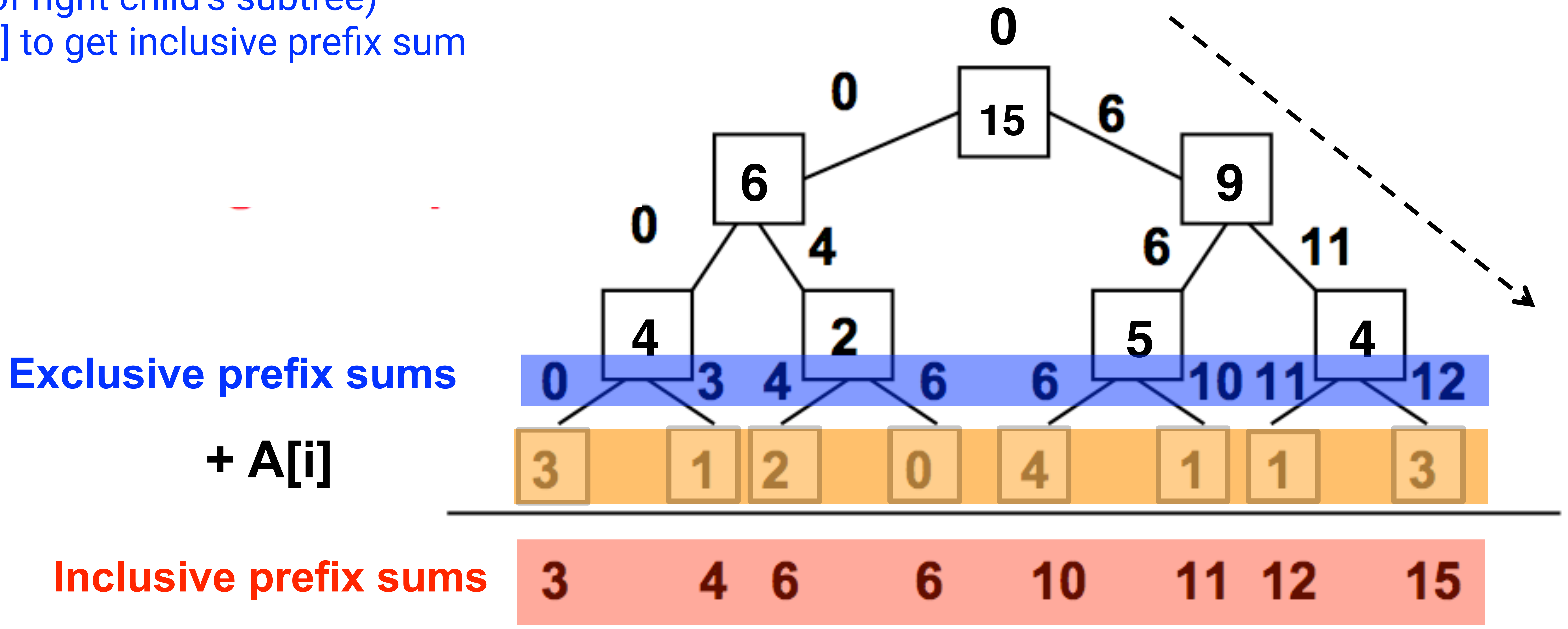
Parallel Prefix Sum: Upward Sweep (while calling scan recursively)

1. Upward sweep is just like Parallel Reduction, except that partial sums are also stored along the way
2. Receive values from left and right children
3. Compute left+right and store in box
4. Send left+right value to parent



Parallel Prefix Sum: Downward Sweep (while returning from recursive calls to scan)

1. Receive value from parent (root receives 0)
2. Send parent's value to LEFT child (prefix sum for elements to left of left child's subtree)
3. Send parent's value+ left child's box value to RIGHT child (prefix sum for elements to left of right child's subtree)
4. Add A[i] to get inclusive prefix sum



Summary of Parallel Prefix Sum Algorithm

- Critical path length, CPL = O(log n)
- Total number of add operations, WORK = O(n)
- Optimal algorithm for P = O(n/log n) processors
  - Adding more processors does not help
- Parallel Prefix Sum has several applications that go way beyond computing the sum of array elements
  - Parallel Prefix Sum can be used for any operation that is associative (need not be commutative)
    - In contrast, finish accumulators required the operator to be both associative and commutative

Parallel Filter Operation

[Credits: David Walker and Andrew W. Appel (Princeton), Dan Grossman (U. Washington)]

Given an array **`input`**, produce an array **`output`** containing only elements such that **`f(elt)`** is true, i.e., `output = input.parallelStream().filter(f).toArray()`

Example: **`input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]`**

```
f: is elt > 10
output [17, 11, 13, 19, 24]
```

Parallelizable?

- Finding elements for the output is easy
- But getting them in the right place seems hard

Parallel prefix to the rescue

1. Parallel map to compute a bit-vector for true elements (can use Java streams)

```
input   [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits    [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector (not available in Java streams)

Parallel prefix to the rescue

1. Parallel map to compute a bit-vector for true elements (can use Java streams)

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector (not available in Java streams)

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. Parallel map to produce the output (can use Java streams)

```
output [17, 11, 13, 19, 24]
```

Parallel prefix to the rescue

1. Parallel map to compute a bit-vector for true elements (can use Java streams)

```
input   [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits    [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector (not available in Java streams)

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. Parallel map to produce the output (can use Java streams)

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){
  if(bits[i]==1)
    output[bitsum[i]-1] = input[i];
}
```

Parallelizing Quicksort (Remember Homework 3?)

	Best / expected case *work*
1. Pick a pivot element	O(1)
2. Partition all the data into:	O(n)
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	2T(n/2)

Simple approach: Do the two recursive calls in parallel

- Work: unchanged at $O(n \texttt{ log } n)$
- Span: now $CPL(n) = O(n) + CPL(n/2) = O(n)$
- So parallelism (i.e., work / span) is $O(\texttt{log } n)$

Sophisticated approach: use scans for the partition step

- Work: unchanged at $O(n \texttt{ log } n)$
- Span: now $CPL(n) = O(\texttt{log } n) + CPL(n/2) = O(\texttt{log}^2 n)$
- So average parallelism (i.e., work / span) is $O(n / \texttt{log } n)$

Use of Prefix Sums to parallelize partition() in Quicksort

```
1.  partition(int[] A, int M, int N) { // choose pivot from M..N
2.   forall (point [k] : [0:N-M]) { // parallel loop
3.    lt[k] = (A[M+k] < A[pivot] ? 1 : 0);  // bit vector with < comparisons
4.    eq[k] = (A[M+k] == A[pivot] ? 1 : 0); // bit vector with = comparisons
5.    gt[k] = (A[M+k] > A[pivot] ? 1 : 0);  // bit vector with > comparisons
6.    buffer[k] = A[M+k];                   // Copy A[M..N] into buffer
7.   }
8.   ... Copy lt, eq, gt, into ltPS, eqPS, gtPS before step 9 ...
9.   final int ltCount = computePrefixSums(ltPS); //update lt with prefix sums
10. final int eqCount = computePrefixSums(eqPS); //update eq with prefix sums
11. final int gtCount = computePrefixSums(gtPS); //update gt with prefix sums
12. // Parallel move from buffer into A
13. forall (point [k] : [0:N-M]) {
14.   if(lt[k]==1) A[M+ltPS[k]-1] = buffer[k];
15.   else if(eq[k]==1) A[M+ltCount+eqPS[k]-1] = buffer[k];
16.   else A[M+ltCount+eqCount+gtPS[k]-1] = buffer[k];
17.  }
18.  . . .
19.}  // partition
```

Example

- Step 1: pick pivot as median of three

8	1	4	9	0	3	5	2	7	6

- Steps 2: implement partition step as two filter/pack operations that store result in a second array

1	4	0	3	5	2				

1	4	0	3	5	2	6	8	9	7

- Step 3: Two recursive sorts in parallel

Example Applications of Parallel Prefix Algorithm

- Prefix Max with Index of First Occurrence: given an input array A, output an array X of objects such that X[i].max is the maximum of elements A[0…i] and X[i].index contains the index of the first occurrence of X[i].max in A[0…i]

- Filter and Packing of Strings: given an input array A identify elements that satisfy some desired property (e.g., uppercase), and pack them in a new output array. (First create a 0/1 array for elements that satisfy the property, and then compute prefix sums to identify locations of elements to be packed.)
  - Useful for parallelizing partitioning step in Parallel Quicksort algorithm